Sudoku - A Tutorial

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5th November 2011

Contents

1 Sudoku Patterns and Sudokus

1.1 Definitions

Definition 1 (Sudoku Pattern) *For* $n = b^2$ $(b = 1, 2, 3, \ldots)$, *let* P *be an* $(n \times n)$-*matrix. Then* P *can be partitioned into* n $(b \times b)$-*matrices called* boxes *(or* subgrids*).*

A sudoku pattern of size $n \times n$ *is an* $(n \times n)$-*matrix the elements of which are any of the digits* $0, 1, \ldots, n$, *satisfying the following 3 conditions (the* B-R-C-conditions*):*

(B) In each box (or subgrid), none of the digits $1, 2, \ldots, n$ *appears more than once (whereas the digit 0 may appear arbitrarily often).*

(R) In each row, none of the digits $1, 2, \ldots, n$ *appears more than once (whereas the digit 0 may appear arbitrarily often).*

(C) In each column, none of the digits $1, 2, \ldots, n$ *appears more than once (whereas the digit 0 may appear arbitrarily often).*

Definition 2 (Complete Sudoku Pattern) *A sudoku pattern is called* complete *if 0 does not occur in it.*

Definition 3 (Extension) *For patterns* P *and* Q, Q *is called* an extension of P*, if the two patterns coincide in all cells of* P *with positive values.*

Definition 4 (Sudoku) *A sudoku pattern* P *is called a* sudoku*, if there exists a pattern* Q *which is complete and an extension of* P*.*

1.2 Remarks

- In this script, the zeros are omitted in order to increase readability.
- From these definitions, it follows that every complete sudoku pattern is a sudoku.
- Of cours, the digits $0, 1, 2, \ldots, n$ can be replaced by any collection of $n+1$ distinct signs, i.e. by a blank for 0 and some of the letters A, B, ...

Example 1.1 (Size 1×1**)**

There are exactly two sudoku patterns, and exactly two sudokus, namely [ ] and [1].

Example 1.2 (Size 4×4**)**

The following two matrices (zeros omitted) are both sudoku patterns, but only the one to the right is a sudoku:

sdk4_00 0

1	2	3	4
3	4	1	2
2	1	4	3
4	3	2	1

sdk4_06 16

Example 1.3 (Size 9×9)

The following two matrices (zeros omitted) are both sudoku patterns, but only the one to the right is a sudoku:

sdk 1

1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
4	**5**	**6**	**7**	**8**	**9**	**1**	**2**	**3**
7	**8**	**9**	**1**	**2**	**3**	**4**	**5**	**6**
2	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**1**
5	**6**	**7**	**8**	**9**	**1**	**2**	**3**	**4**
8	**9**	**1**	**2**	**3**	**4**	**5**	**6**	**7**
3	**4**	**5**	**6**	**7**	**8**	**9**	**1**	**2**
6	**7**	**8**	**9**	**1**	**2**	**3**	**4**	**5**
9	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**

sdk 2

1.3 All sudokus of size 4×4

There are exactly 288 complete sudokus of size 4×4.

Proof: We suppose the first (upper left) box to be filled as shown below:

1	2		
3	4		

`sdk4_01` 4

Then in box 2 (upper right) the first row must contain 3 and 4, and the second row must contain 1 and 2. Therefore, there remain 4 possible cases:

1	2	3	4
3	4	1	2

`sdk4_02` 8

1	2	3	4
3	4	2	1

`sdk4_03` 8

1	2	4	3
3	4	1	2

`sdk4_04` 8

1	2	4	3
3	4	2	1

`sdk4_05` 8

Case I: Let's consider box 3 (lower left). The first column must be formed from of 2 and 4, and the second from 1 and 3. In each of the two columns, the digits can be permuted independently, each state of box 3 uniquely determining box 4. Therefore, there are 4 possibilities.

Case II: In box 3, the rows $(2, 3)$ and $(4, 1)$ lead to a contradiction (i.e. there would be no complete extension). There remain, for rows 3 and 4 of the full pattern, just the two possibilities $(2, 1, 4, 3)$ and $(4, 3, 1, 2)$. Interchanging these two rows gives rise to 2 possibilities.

Case III: This case is quite analogous to case II. There are 2 possibilities.

Case IV: This case is analogous to case I. There are 4 possibilities.

So the total number of possibilities amounts to $4 + 2 + 2 + 4 = 12$. All other possibilities can be obtained by a permutation of the digits $1, 2, 3, 4$. The total number of permutations is $4! = 24$. Therefore, there are

$$4! \cdot (4 + 2 + 2 + 4) = 288$$

complete 4×4 sudokus. Q.E.D.

1.4 About the sudokus of size 9×9

How many complete sudokus are there of size 9×9? Independently from each other, Bertram Felgenhauer and Frazer Jarvis [1] have found this number to be

$$6670\ 903752\ 021072\ 936690 \approx 6.671 \cdot 10^{21}.$$

If you consider sudokus equivalent if they can be mapped to each other by a permutation of the digits 1 through 9, the number of distinct complete sudokus still amounts to $18383\ 222420\ 692992 \approx 1.838 \cdot 10^{16}$.

Example 1.4 (Minimal sudoku)

The next sudoku pattern is a sudoku, i.e. has exactly one completion.

	1							**9**
			3			**8**		
						6		
				1	**2**	**4**		
7		**3**						
5								
8			**6**					
				4			**2**	
			7				**5**	

sdk 3

It has only 17 clues (given digits). At the moment (5th November 2011), there are no known sudokus with less than 17 clues.

Example 1.5 (Sudoku patterns, but no sudokus)

Of the following three sudoku patterns, none is a sudoku. The one to the left has 85 distinct completions. The one in the middle is the common part (the intersection) of all completions. It has the same completions as the original pattern. The one to the right differs from the original one only in that there is a 2 in cell $(2, 6)$. It is therefore a sudoku pattern with no completion at all, as all completions have a 1 in cell $(2, 6)$. Sudoku patterns are no sudokus if and only if the have no completion at all, or more than one completion.

		3		**9**		**6**	**4**	
	1		**7**		**8**		**2**	
		9				**4**		
	8		**4**		**5**		**9**	
		5				**8**		
	6		**8**		**2**		**5**	
	4	**1**		**6**		**2**	**3**	

sdk 4

		3		**9**	**1**	**6**	**4**	
	1		**7**		**8**		**2**	
		9		**8**		**4**		**5**
	8		**4**		**5**		**9**	
4		**5**				**8**		
	6	**7**	**8**		**2**		**5**	
	4	**1**		**6**		**2**	**3**	

sdk 5

		3		**9**	**2**	**6**	**4**	
	1		**7**		**8**		**2**	
		9		**8**		**4**		**5**
	8		**4**		**5**		**9**	
4		**5**				**8**		
	6	**7**	**8**		**2**		**5**	
	4	**1**		**6**		**2**	**3**	

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1.5 Completion by trial and error

Every sudoku can be completed, in theory, without any rules, just by trial and error. Take an empty cell and substitute in turn every digit which is not barred. Iterate the procedure until all the extensions but one cease to be sudoku patterns because they violate the B-R-C-conditions. The surviving one is the desired completion. This method is, of course, extremely laborious (and boring). The following rules make sudoku completion more exciting.

1.6 Constraint propagation

The aim of this paper is to present the *propagation rules* of sudoku completion as strictly, and as briefly as possible. We stick to the notions and denotations as used in Fowler[2]. The propagation rules are:

F Uniqueness of a digit in a given cell (F: "Field")

N Uniqueness of a cell for a given digit in a given box/row/column (N: "only")

B Box - row / Box - column interactions

T Use of naked and hidden tuples (T: "Tupel")

X X-chains (one-candidate chains)

Y Y-chains (pair chains)

W W-patterns (x-wing, swordfisch)

These rules can be applyed to sudokus of any size. However, we now focus on size 9×9.

2 Elementary Sudokus (Rules F and N)

2.1 The elementary rules

Definition 5 (Digits barred from cells) *We say that a given digit is barred from a given empty cell if the digit already occupies a cell in the same box, or the same row, or the same column.*

Rule 1 (F: Field) *If from a given cell (field) all digits are barred but one, put this digit into the cell.*

Rule 2 (N: oNly cell) *There are three subrules:*

N_B **Box scanning for a digit** *If in a given box, a digit is barred from all empty cells but one, put the digit into this cell.*

N_R **Row scanning for a digit** *If in a given row, a digit is barred from all empty cells but one, put the digit into this cell.*

N_R **Column scanning for a digit** *If in a given column, a digit is barred from all empty cells but one, put the digit into this cell.*

We might be tempted to put it shorter, and just say, for example: If in a given box, a given digit is possible in just one cell, put the digit there. But this would be a misleading rule. In any sudoku (but not in any sudoku *pattern*), there is just one digit possible in any cell.

Definition 6 (Elementary rules, elementary sudokus) *By the* elementary rules, *we understand the rules F, N_B, N_R, and N_C. A sudoku is* elementary, *if and only if it allows completion by the elementary rules.*

2.2 Independence of F, N_b, N_r, and N_c

The four rules are independent from each other in the sense that none of them can be replaced with a combination of the three others. We give four sudoku patterns each one of which can be extended by one, and only one, of the four rules.

Example 2.1 (Only F)

	2	3		9		6	4	
	1		7		8		2	
		9		8		4		5
	8		4		5		9	
4		5				8		
	6		8		2		5	
	4	1		6		2	3	

sdk 7

$= F \Rightarrow$

	2	3		9	1	6	4	
	1		7		8		2	
		9		8		4		5
	8		4		5		9	
4		5				8		
	6	7	8		2		5	
	4	1		6		2	3	

sdk 8

- For cell (2,6), there is only one possible digit, namely 1.
- For cell (7,3), there is only one possible digit, namely 7.

Example 2.2 (Only N_B)

	1	2	8	4				
	9	3				4		
	4				3			
1				3	4			
	5					1	3	4
			1	7				9
			6				2	
		1	3			7	4	
			4		5	8	1	

sdk 9

$= N_B \Rightarrow$

	1	2	8	4				
	9	3				4		
	4				3			
1			5	3	4			
	5					1	3	4
			1	7				9
			6		7	9	2	
		1	3			7	4	
			4		5	8	1	

sdk 10

- In box $B_{2,2}$ (center), digit 5 can only be placed in cell (4,4).
- In box $B_{3,2}$, digit 7 can only be placed in cell (7,6).
- In box $B_{3,3}$, digit 9 can only be placed in cell (7,7).

Example 2.3 (Only N_R)

				4	8	9		
	8	4	2			5	1	
	9		3			2	8	4
9						6	4	
	4	5				7		
	3	8			4	1		2
4	6				1	8	2	
8	2		4		5	3		
		9	8	2		4		

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$= N_R \Rightarrow$

				4	8	9		
	8	4	2			5	1	
	9		3			2	8	4
9						6	4	
	4	5				7		
	3	8			4	1		2
4	6				1	8	2	5
8	2		4		5	3		
		9	8	2		4		

sdk 12

- In row 7, digit 5 is only possible in the last column.

Example 2.4 (Only N_C)

					1	2		3
	4			5			7	
								6
1					7			
	8						9	
			3					2
5		8					2	
				9			8	
3		2	6					

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$= N_C \Rightarrow$

					1	2		3
	4			5			7	
								6
1					7		3	
	8						9	
			3					2
5		8					2	
				9			8	
3		2	6					

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- In column 8, digit 3 can only be placed in the 4th row.

2.3 Most easy to complete (rule N_b)

If a sudoku is completed with the aid of a candidate list, then the most convenient rule is of course F. However, if the sudoku is completed "on sight", i.e. without any auxiliary notes, the easiest way to extend sudoku patterns is by rule N_B. Here is an example:

8→(4,2)	6→(8,4)	5→(2,5)	1→(5,9)
7→(5,6)	5→(8,9)	3→(2,9)	4→(2,4)
2→(8,5)	3→(9,8)	6→(4,3)	9→(3,8)
5→(5,2)	1→(6,2)	2→(5,8)	1→(4,5)
3→(5,4)	2→(6,3)	4→(8,7)	9→(6,5)
2→(7,2)	3→(6,7)	3→(1,2)	4→(1,8)
8→(8,1)	5→(7,4)	8→(1,5)	9→(2,6)
5→(4,8)	1→(7,8)	1→(2,7)	4→(3,2)
8→(7,6)	7→(8,8)	6→(6,8)	6→(2,2)
9→(8,2)	7→(9,2)	8→(2,8)	6→(3,6)
3→(8,6)	4→(9,5)	1→(3,4)	
9→(5,1)	2→(2,1)	9→(4,7)	
1→(8,3)	7→(2,3)	6→(5,5)	

1		9	7		2	5		6
5		8		3		7		2
3			2		4			7
		4				8		
7			8		5			4
4		3		7		6		9
6		5	9		1	2		8

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The protokoll on the left shows how to proceed.

2.4 Iteration of the elementary rules

Iterating the four elementary rules (ER) does not, in general, lead to a completion of a given sudoku. Some examples:

Example 2.5 (No elementary completion)

Iteration of the four basic methods stops when 31 digits are determined. None of the four methods can add more digits.

3	4		6					
		7						
	2			8		5	7	
					5			
	7			1			2	
			4					
	3	6		2			1	
						9		
					7		8	2

sdk 16

= (ER) ⇒

3	4	5	6	7	1	2	9	8
		7	2	5				
	2			8		5	7	
			7		5			
	7			1			2	
			4		2			
	3	6		2			1	
						9		
					7		8	2

sdk 17

Example 2.6 (Elementary completion)

			4	9		2		7
	4		1			9		
				7	2			1
	8					4		2
7								5
6		2					1	
4			2	6				
		8			5		6	
5		3		8	7			

sdk 18

= (ER) ⇒

8	5	1	4	9	6	2	3	7
2	4	7	1	3	8	9	5	6
9	3	6	5	7	2	8	4	1
3	8	5	6	1	9	4	7	2
7	1	4	8	2	3	6	9	5
6	9	2	7	5	4	3	1	8
4	7	9	2	6	1	5	8	3
1	2	8	3	4	5	7	6	9
5	6	3	9	8	7	1	2	4

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In this example, completion can even be attained by an iteration of rule N_B.

2.5 Problems

The following three sudokus can be completed by any one of the four elementary rules:

8		5		9		7		2
3		4	2		7	6		5
		9	7		1	2		
1								8
		2	8		5	1		
5		6	3		4	8		7
9		1		2		3		6

sdk 20

6								3
			7	6	3			
		5				8		
	3			5			4	
	4	2	8		1	5	9	
	6			2			7	
		3		9		6		
			1	7	4			
4								1

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4	3		5		7		2	9
5								7
			1	2	9			
2		3				5		4
		9				2		
1		7				6		3
			3	5	6			
3								8
9	1		2		4		3	6

sdk 22

The following 9 sudokus can be completed by rule N_B alone:

		2		1				6
				9				3
	8				4	9		
	4		7					
7			9		1			8
					3		6	
		3	4				9	
1				7	8			
6				2		4		

sdk 23

6								3
			7	6	3			
		5		4		8		
	3			5			4	
	4	2	8		1	5	9	
	6			2			7	
		3		9				
			1	7	4			
4								1

sdk 24

9		3				5		6
			3		1			
5								2
	4			9			8	
			4		2			
	8			6			1	
7								4
			5		7			
1		2				8		5

sdk 25

1			2	9				7
					6			
	3		7				5	
	1			4		7		3
2								6
6		4		5			8	
	4				1		7	
			3					
9				6	4			2

sdk 26

1			8			9		
	4			1			6	
		2			5			7
8			3			7		
	1			2			5	
		9			7			4
6			5			1		
	9			6			4	
		1			2			8

sdk 27

9								5
	8		6		7		2	
			9		2			
	4	8		9		7	1	
			7		5			
	3	7		1		5	8	
			1		9			
	9		5		4		7	
1								2

sdk 28

	5		3	8		9	4	
4			6		1	5		8
7	8							
	2						3	7
5								9
8	3						6	
							5	1
3		1	2		4			6
	6	5		1	3		2	

sdk 29

6		5	1		8	3		2
9		2				7		5
3				7				8
			6		1			
7				5				1
5		3				6		7
2		1	7		3	5		4

sdk 30

			4			5		
		2		5				
	7		2			8		9
4		6		3				
	3		1		9		8	
				8		3		4
3		5			1		2	
				7		1		
		9			3			

sdk 31

The following three sudokus can be completed by rule N_R alone:

5				1				8
			2	6	9			
		6				9		
	9			5			1	
7	1		8		4		3	5
	6			7			2	
		1				4		
			4	2	1			
9				3				6

sdk 32

	5			4			9	
7		1						2
	8				3			
			3		7	5		
8								6
		6	9		8			
			7				2	
5						9		3
	6			1			7	

sdk 33

1			9		7	5		4
3		4	8			9		
9				4		3		2
7		5		6				1
		9			2	1		6
4		8	7		3			9

sdk 34

The following three sudokus can be completed by rule N_C alone:

8	9							5
				6		7		4
	5				1			
		8	6		4			
	1			5			6	
			2			1		
			7				8	
5		6		3				
9							2	7

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			8					
9					4			7
3	5				2		4	9
	3	8						5
				7				
4						2	6	
2	1		5				7	3
7			2					4
					6			

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	5							
7	1	9		4				
	8				7			2
	9			3				6
	2	1	9		4	3	8	
5				6			9	
8			7				4	
				8		7	5	9
							3	

sdk 37

The following three sudokus can be completed by rule F alone:

3		1	2	5		8		4
		6	8			5		
		9				6		8
2								7
8		5				1		
		8	4		2	9		
7		3		8	5	4		2

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		9	1		6	7		
				8				
3								2
2			7		3			8
	5			4			9	
1			2		8			4
4								7
				6				
		3	4		5	1		

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	2							
		7	8		6	1		2
				2			3	
	9			3			6	
		4				7		
	3			1			4	
	1			4			7	
6		2	3		9	8		
							5	

sdk 40

The following six sudokus need for completion at least three of the elementary rules, the two last ones need all four:

		5						4
		7			5	2		
3	2			9	7		5	
							3	
		1		2		9		
	5	9						
	9		6	7			4	2
		2	8			1		
5						6		

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9								6
		7	5	3	4			
							5	
							4	
	1		9	8	2		6	
	3						8	
	4							
			7	4	1	2		
8								7

sdk 42

			7			2	5	
		8						6
	9			3				8
5				2				
		3	9		4	8		
				6	7			1
7				1			4	
3						6		
	1	6			2			

sdk 43

		8			6		2	
		6	9					4
1	4			7				
	2							1
		5				3		
4							8	
				6		7	3	5
3					5	4		
	9		1			8		

sdk 44

		9					4	6
				3	7	8		
					3	1	2	
				8				
	5	4	6					
		8	4	1				
2	9					7		

sdk 45

	3						5	7
				3	7	9	6	
4	7				1	2		
5	2				8			
		4			3			5
6		3	1					
8							9	
	4			9	5			1
7							4	

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3 Candidate Tables

For a given sudoku we obtain a *candidate table* by writing down, for each cell, a set of digits such that this set contains the digit which occupies the cell in the completion.

By this definition, a candidate table can be very redundant; it may attribute to every cell the full set of positive digits $1, 2, 3, \ldots, 9$. But typically, we start out with a candidate table in which all the barred elements are omitted. Completing a sudoku means to reduce the candidate table up to the point where to each cell just one candidate is attributed.

3.1 Candidate tables and elementary rules

Although elementary sudokus can comfortably be completed without auxiliary notes, the elementary rules can be illustrated with candidate tables.

1234578	1578	12347	4579	346	3679	134689	4569	1345689
3457	**9**	347	**8**	346	**1**	346	**2**	3456
13458	158	**6**	459	**2**	39	**7**	459	134589
1467	**3**	147	127	**9**	27	146	**8**	14567
1789	178	**5**	**6**	138	**4**	**2**	79	179
146789	**2**	1479	17	**5**	78	1469	**3**	14679
12369	16	**8**	1249	**7**	269	**5**	469	23469
2679	**4**	279	**3**	68	**5**	689	**1**	26789
1235679	1567	12379	1249	1468	2689	34689	4679	2346789

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126	**5**	26	**3**	**8**	27	**9**	**4**	2
4	9	239	**6**	279	**1**	**5**	7	**8**
7	**8**	2369	459	2459	259	1236	1	23
169	**2**	469	14589	4569	5689	148	**3**	**7**
5	147	467	1478	23467	2678	1248	18	**9**
8	**3**	479	14579	24579	2579	124	**6**	245
29	479	24789	789	679	6789	3478	**5**	**1**
3	79	**1**	**2**	579	**4**	78	789	**6**
9	**6**	**5**	789	**1**	**3**	478	**2**	4

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Rule F In the sudoku on the left, there is no cell with a single candidate. Therefore, this rule has no effect. In the sudoku on the right, however, there are 6 such cells. Therefore, rule F allows us to immediately expand the sudoku by 6 more digits.

Rule N_B In the sudoku on the left, this rule leads to two more final digits: 3 in box $B_{2,2}$, 5 in box $B_{2,3}$. In the sudoku at the right, this rule leads to 9 more final digits: 1 in box $B_{1,1}$, 6 and 7 in box $B_{1,3}$, 3 in box $B_{2,2}$, 5 in box $B_{2,3}$, 8 in box $B_{3,1}$, 5 in box $B_{3,2}$, 3 and 9 in box $B_{3,3}$.

Rule N_R In the sudoku on the left, this rule leads to two final digits: 5 in cell $(4, 9)$, and 3 in cell $(5, 5)$. In the sudoku at the right, this rule leads to 6 more final digits: 1 in cell $(1, 1)$, 7 in cell $(1, 6)$, 3 in cell $(2, 3)$, 3 in cells $(5, 5)$ and $(7, 7)$, and 5 in cell $(8, 5)$.

Rule N_C In the sudoku on the left, this rule leads has no effect. In the sudoku at the right, it leads to to 7 more final digits: 1 in cell $(5, 2)$, 8 in cell $(7, 3)$, 3 in cell $(5, 5)$, 6 in cell $(3, 7)$, 9 in cell $(8, 8)$, 3 in cell $(3, 9)$, 5 in cell $(6, 9)$,

The rules may overlap. In the sudoku on the right, for instance, candidate 3 in cell $(5, 5)$ is the final digit by rule N_B, as well as by rules N_R and N_C (but not by F).

If we apply, for instance, rule N_B to both sudokus, we get

	9		**8**		**1**		**2**	
		6		**2**		**7**		
	3			**9**			**8**	**5**
		5	**6**	**3**	**4**	**2**		
	2			**5**			**3**	
		8		**7**		**5**		
	4		**3**		**5**		**1**	

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1	**5**		**3**	**8**		**9**	**4**	
4			**6**		**1**	**5**	**7**	**8**
7	**8**					**6**		
	2						**3**	**7**
5				**3**				**9**
8	**3**						**6**	**5**
		8				**3**	**5**	**1**
3		**1**	**2**	**5**	**4**		**9**	**6**
	6	**5**		**1**	**3**		**2**	

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4 Box-Row and Box-Column Interactions

4.1 Rule B and its 4 subrules

Rule $B \succ R$ If within some box, the candidates of a given digit are restricted to one single row, all further candidates of the digit that occur within this row but outside the given box can be eliminated.

Rule $B \succ C$ If within some box, the candidates of a given digit are restricted to one single column, all further candidates of the digit that occur within this column but outside the given box can be eliminated.

Rule $R \succ B$ If within some row, the candidates of a given digit are restricted to one single box, then they can be eliminated from the other two rows of the box.

Rule $C \succ B$ If within some column, the candidates of a given digit are restricted to one single box, then they can be eliminated from the other two columns of the box.

The following examples show sudokus which can be completed almost by elementary rules alone. Just once in each case, one of the B rules is necessary.

Example 4.1 (Rule $B \succ R$)

	9				8	1	6	
		1	7			9	2	
2								3
3		8	4		1	5		6
	4	9	8			2	3	
6					7	8		4
8			6	4				9
				7	2	6		
	6	7	1	8			5	2

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4 5 7	**9**	3 4 5	2 3 5	2 3 5	**8**	**1**	**6**	5 7
4 5	3 5 8	**1**	**7**	3 5 6	3 4 5 6	**9**	**2**	5 8
2	5 7 8	**6**	5 9	**1**	4 5 9	4 7	4 7 8	**3**
3	2 7	**8**	**4**	2 9	**1**	**5**	7 9	**6**
1 5 7	**4**	**9**	**8**	5 6	5 6	**2**	**3**	1 7
6	1 2 5	2 5	2 3 5 9	2 3 5 9	**7**	**8**	1 9	**4**
8	1 2 3 5	2 3 5	**6**	**4**	3 5	3 7	1 7	**9**
1 4 5 9	1 3 5	3 4 5	3 5 9	**7**	**2**	**6**	1 4 8	1 8
4 9	**6**	**7**	**1**	**8**	3 9	3 4	**5**	**2**

sdk 50

By twice applying rule N_B, the sudoku on the left is extended to the sudoku on the right. Then in box $B_{1,3}$, candidate 4 is restricted to row 3. Therefore by rule $B \succ R$, candidate 4 can be eliminated in cell $(3, 6)$. As a consequence, 4 is put into cell $(2, 6)$ by rule N_B. Then completion can be achieved by rule F alone.

Example 4.2 (Rule $R \succ B$)

			5				2	
1	8			4				
		5					6	
		6	1					4
		2		3		8		
3					7	9		
	5					7		
				2			1	5
	7				4			

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4 6 9	3 6	3 4 7 9	**5**	1 7 9	1 3 8	1 3 4	**2**	3 7 8
1	**8**	3 7	**6**	**4**	**2**	**5**	**9**	3 7
4 9	**2**	**5**	3 8 9	1 7 9	1 3 8	1 3 4	**6**	3 7 8
7	**9**	**6**	**1**	**8**	**5**	**2**	**3**	**4**
5	**1**	**2**	**4**	**3**	**9**	**8**	**7**	**6**
3	**4**	**8**	**2**	**6**	**7**	**9**	**5**	**1**
2 6 8 ~~(9)~~	**5**	3 ~~(9)~~	3 8 9	1 9	1 3 6 8	**7**	**4**	2 3 9
4 6 8 (9)	3 6	3 4 (9)	**7**	**2**	3 6 8	3 6	**1**	**5**
2 6 ~~(9)~~	**7**	**1**	3 9	**5**	**4**	3 6	**8**	2 3 9

sdk 52

Again, the sudoku on the left is extended to the sudoku on the right by elementary rules alone. Then in row 8, candidate 9 is restricted to box $B_{3,1}$. By rule $R \succ B$, all other instances of candidate 9 can be eliminated in this box. Then completion can be achieved by rule F alone.

Example 4.3 (Rule $B \succ C$)

				3	1	6	4	
5					6	2	9	
6								
	1		7					
		4				1		
					8		6	
								9
	9	7	4					2
	2	5	3	7				

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2 7 ~~(8)~~ 9	7 8	2 8 9	2 5 9	**3**	**1**	**6**	**4**	5 8
5	**3**	**1**	**8**	**4**	**6**	**2**	**9**	**7**
6	**4**	2 8 9	2 5 9	2 5 9	**7**	3 5 8	3 5 8	**1**
2 ~~(8)~~ 9	**1**	**6**	**7**	2 5 9	**4**	5 8 9	2 5 8	**3**
2 7 ~~(8)~~ 9	5 7 8	**4**	**6**	2 5 9	**3**	**1**	2 5 7 8	5 8
3	5 7	2 9	2 5 9	**1**	**8**	5 7 9	**6**	**4**
4	**6**	**3**	**1**	**8**	**2**	5 7	5 7	**9**
1 (8)	**9**	**7**	**4**	**6**	**5**	3 8	1 3 8	**2**
1 (8)	**2**	**5**	**3**	**7**	**9**	**4**	1 8	**6**

sdk 54

Again, the sudoku on the left is extended to the sudoku on the right by elementary rules alone. Then in box $B_{3,1}$, candidate 8 is restricted to column 1. By rule $B \succ C$, the other instances of candidate 8 can be eliminated in column 1. Then by rule N_B, cell $(5, 2)$ has to be set to 8. Completion can now be achieved by rules F and N_B.

In the first and the last of these three examples, candidate tables are not really needed. In many cases, rules $B \succ R$ and $B \succ C$ lead, together with an elementary rule, directly to another final digit.

Example 4.4 (Rule $C \succ B$)

		3				**8**		
					1			
6				**7**				**5**
2	**7**		**3**		**8**			
		1		**2**		**4**		
			6		**4**		**7**	**9**
3				**8**				**7**
			4					
		2	**5**			**1**		

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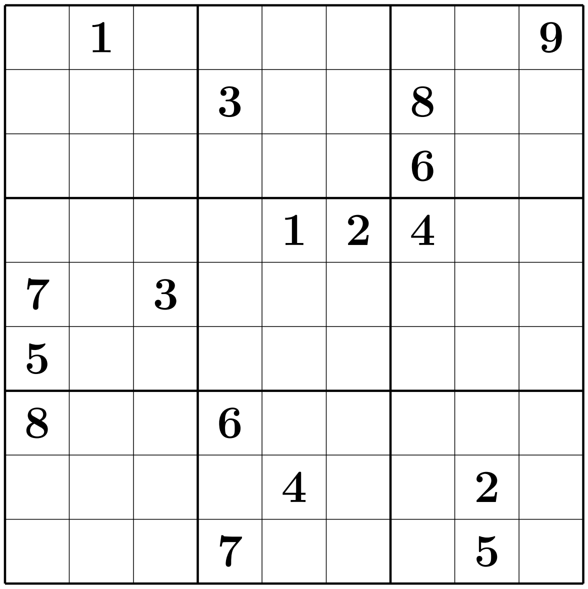
7	1 2 4 5 9	**3**	2 9	4 5 6	2 6 9	**8**	1 2 4 6 9	1 2 4 6
4 5 8	2 4 5 8 9	5 8 9	2 8 9	3 4 5 6	**1**	**7**	2 3 4 6 9	2 3 4 6
6	1 2 4 8 9	8 9	2 8 9	**7**	2 3 9	3 9	1 2 3 4 9	**5**
2	**7**	**4**	**3**	**9**	**8**	5 6	1 5 6	1 6
9	**6**	**1**	**7**	**2**	**5**	**4**	3 8	3 8
5 8	**3**	5 8	**6**	**1**	**4**	**2**	**7**	**9**
3	4 5 9	**6**	**1**	**8**	2 9	5 9	2 4 5 9	**7**
1	5 8 9	**7**	**4**	3 6	2 3 6 9	3 5 6 9	2 3 5 6 8 9	2 3 6 8
4 8	4 8 9	**2**	**5**	3 6	**7**	**1**	3 4 6 8 9	3 4 6 8

sdk 56

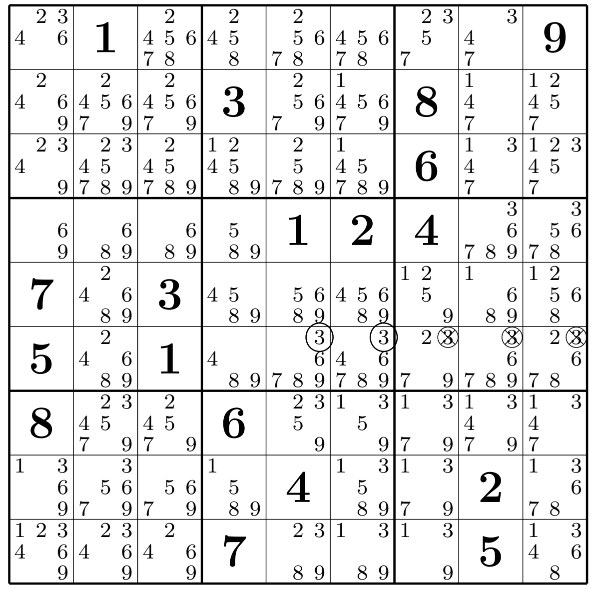
Again, the sudoku on the left is extended to the sudoku on the right by elementary rules. In column 3, candidate 9 is restricted to box $B_{1,1}$. Therefore by rule $C \succ B$, candidate 9 can be eliminated in the rest of this box. In column 4, candidates 2 and 9 are both restricted to box $B_{1,2}$ and can therefore be eliminated in cells $(1, 6)$ and $(3, 6)$.

Example 4.5

We return to the minimal sudoku with just 17 clues from example 1.4. It can be completed solely by the elementary rules $B \succ R$, and $B \succ C$.

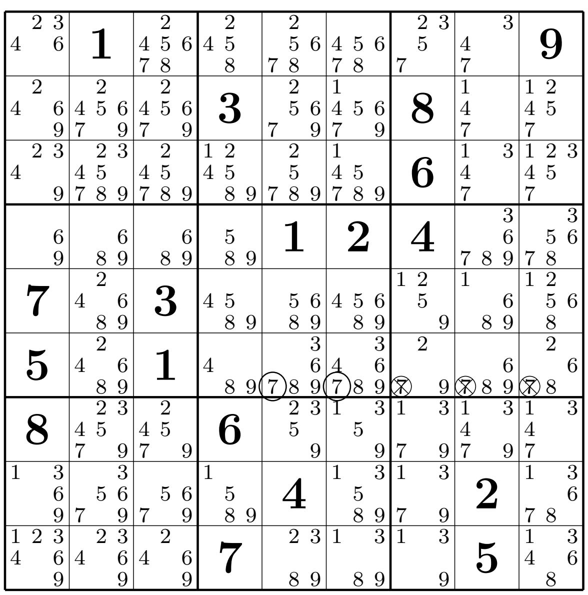


sdk 57

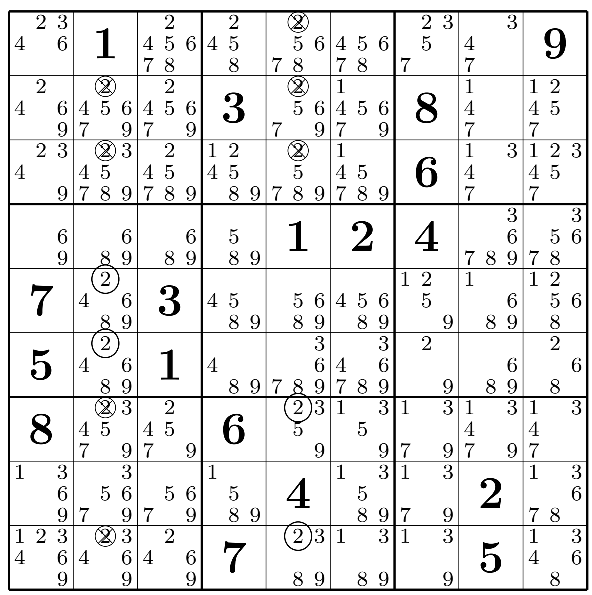


sdk 58

Rule N_B requires cell $(6,3)$ of sudoku 57 to be set to 1. The result is sudoku 58. Then in box $B_{2,2}$, candidate 3 is restricted to row 6. Therefore by rule $B \succ R$, candidate 3 can be eliminated from cells $(6,7)$, $(6,8)$, and $(6,9)$. This leads to sudoku 59.

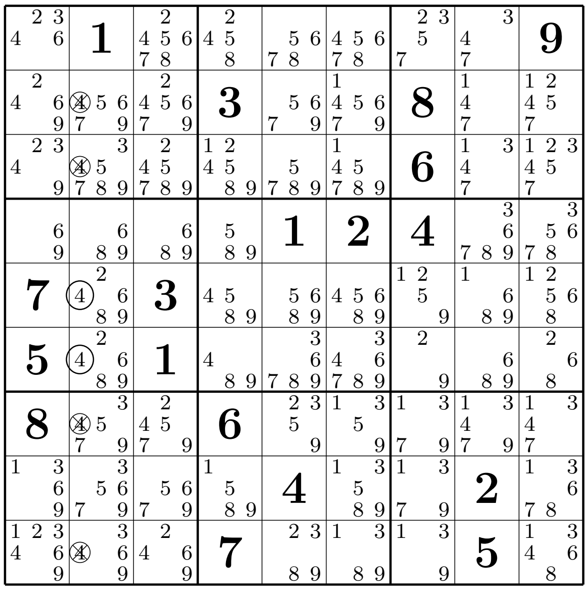


sdk 59

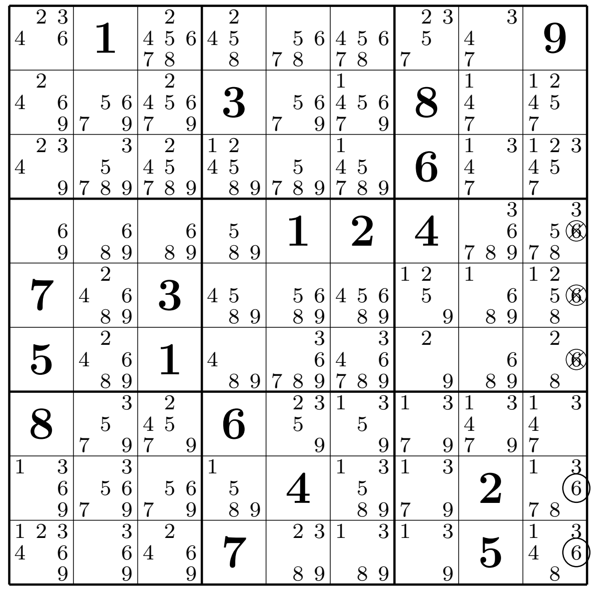


sdk 60

Also in box $B_{2,2}$, candidate 7 is restricted to row 6. Therefore by rule $B \succ R$, candidate 7 can be eliminated from cells $(6,7)$, $(6,8)$, and $(6,9)$. This leads to sudoku 60. Then rule $B \succ C$ can be applied to candidate 2 in column 2 as well as in column 5, and we get sudoku 61.

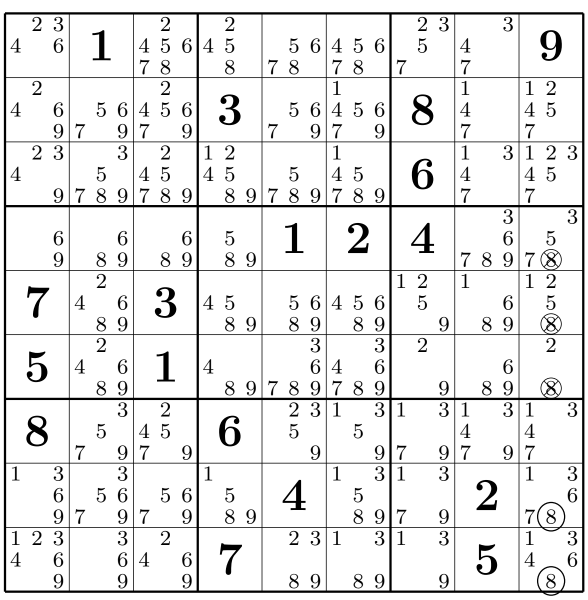


sdk 61

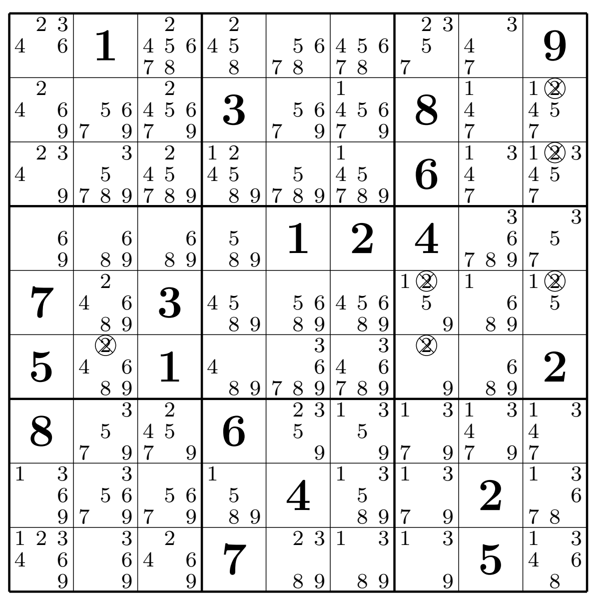


sdk 62

Rule $B \succ C$ can be applied to candidate 4 in column 2 of sudoku 61, and to candidate 6 in column 9 of sudoku 62.

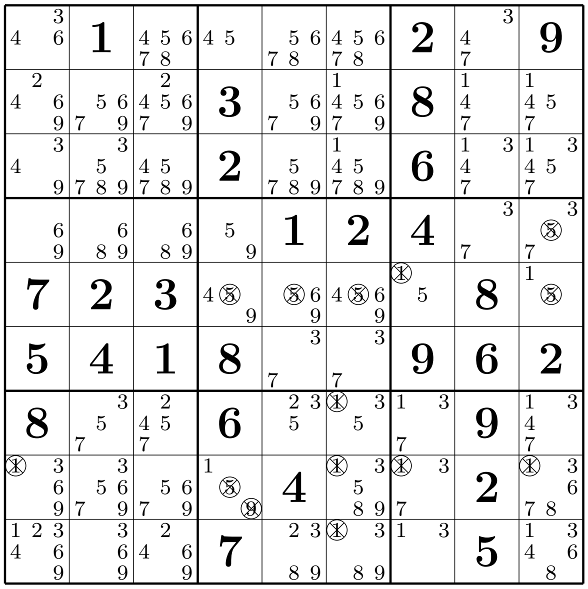


sdk 63

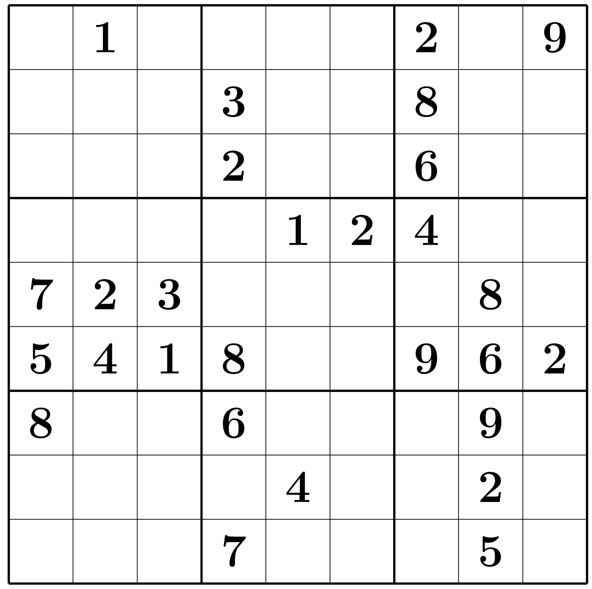


sdk 64

So far, applying rules $B \succ R$ and $B \succ C$ has not resulted in any additional digit, but only in reducing the total number of candidates from 289 to 269. But when we now apply rule $B \succ C$ to candidate 8 in column 9 of sudoku 63, cell $(6, 9)$ has to be put to 2, and we get sudoku 64. Then by repeated applications of elementary rules (F and N_B suffice), we get sudoku 65.



sdk 65



sdk 66

In sudoku 65, cells $(5,7)$ and $(8,4)$ have to be set to 5 and 1, respectively. Therefore, these candidates are already crossed out in the associated cells.

Remark Sudoku 65 is elementary and is obtained from sudoku 64 by elementary rules alone. But sudoku 64 itself is *not* elementary. At first, this seems to be a contradiction. The explanation lies in the fact that the candidate table of sudoku 64 has previously been reduced by non-elementary rules also. It comprises 265 candidates. However, if we started out with the 19 clues (definitely set digits) of sudoku 64 from scratch, we would get a total of 278 candidates in the table. Sudokus 65 and 66 have the same clues. The fact that sudoku 66 is elementary is mirrored by that its candidate table is equal to, and not an extension of, the candidate table of sudoku 65.

4.2 Problems

The next two sudokus can be completed by elementary rules and a single application of rule $B \succ R$:

4		3	6					7
		5	9		8			2
		4				3		6
3		8				1		
2		6	3		7	4		
7					5	8		1

sdk 65

1			2		3			5
		5		9	8	3		
	5	2				7	8	
	6						5	
	9	3				2	1	
		8	9	7	4	5		
6			8		2			9

sdk 66

The next two sudokus can be completed by elementary rules and a single application of rule $B \succ C$:

4					3	2		7
3		5	4		8	6		
6		9				3		
		8				7		5
			3		1	4		8
7		2	6					1

sdk 67

8		9			2			5
	6			4	3			
						8		
				6		5		
6		8				4	2	7
		2		3				
		4						
			1	2			7	
9			5			1		6

sdk 68

The next two sudokus can be completed by elementary rules and a single application of $R \succ B$, and $C \succ B$, respectively:

2	4						3	8
6								2
			2		4	7		
			4	2	9			
		2				6		
			3	5	6			
		9	6		5	2		
8								5
5	3						4	1

sdk 69 Δsdk9_Knaur_121_Z11_trsf

			6	7				
	8					2		5
	4				9			
		5	4					
	1			9			7	
					2	1		
			1				5	
6		7					4	
				8	3			2

sdk 70

The next 6 sudokus can be completed by elementary rules and repeated (less than 10) applications of B rules.

					9	7		
	4			2			3	
8			6					
1						4		
	5		8	4	3		6	
		7						
					6			5
	3			5			2	
		6	9					

sdk 71

7					1		6	3
	3		9			8		
2								
4				2		7		
		9				1		
		2		7				6
								9
		7			6		4	
5	1		8					2

sdk 72

	1	8		2				
					9			4
		4		3		6		5
	3							
7				5		9		3
							1	
2		3		4		8		
5			1					
				7		5	4	

sdk 73

2		7				3		5
					4			
1								4
	7			4			6	
			9	1	7			
	5			3			2	
7								2
			8		6			
9		3				8		7

sdk 74

		2				9		
			9		5			
	3		2		6		8	
	8	4				6	3	
	6	5				8	9	
	5		6		9		2	
			3		8			
		7				4		

sdk 75

		5						4
		9	5				8	
	1		4				9	5
	5	2			7			
				2				
			8			6	7	
3	7				4		2	
	8				1			
2						3		

sdk 76

5 Tuple Reduction

5.1 Open tuples

Definition 7 (Associated cells)

Box association *We say that cells are* box-associated*, or* associated with respect to a box*, if they lie within one and the same box.*

Row association *We say that cells are* row-associated*, or* associated with respect to a row*, if they lie within one and the same row.*

Column association *We say that cells are* column-associated*, or* associated with respect to a column*, if they lie within one and the same column.*

Associated cells *We call cells* associated*, if they are contained in a common box, or a common row, or a common column.*

Suppose that some unoccupied cells are associated with respect to a box, or a row, or a column. Then the number of distinct candidates cannot be smaller than the number of cells, as otherwise, there would be no completion.

Definition 8 (Open tuple) *If in a set of associated unoccupied cells the number of distinct candidates does not exceed the number of cells, we say that the cells form an open tuple. Open tuples are sometimes called* exact *or* naked *tuples.*

Definition 9 (Irreducible open tuple) *By an* irreducible open tuple *we understand an open tuple which does not contain a proper subtuple of cells which is also open.*

Technically speaking, an open tuple t is an ordered pair of two sets of equal size, the first consisting of candidates (digits from 1 through 9), the second of unoccupied associated cells (fields):

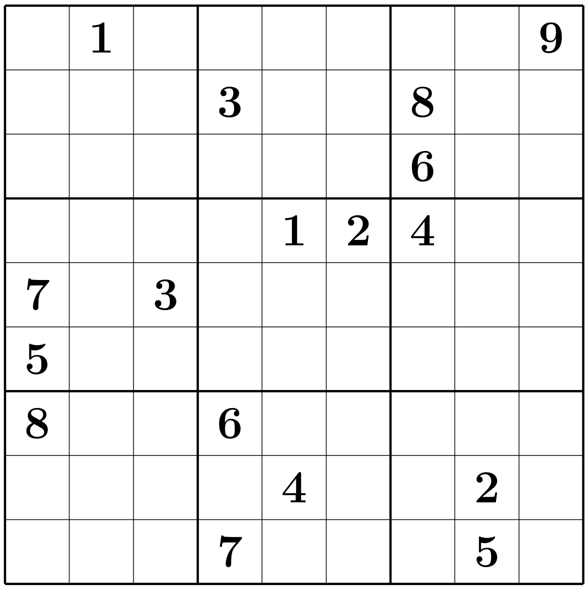
$$t = (\{c_1, \ldots, c_k\}, \{f_1, \ldots, f_k\}).$$

If the cells $f_1, \ldots, f_k$ are box-associated, we say that the candidates $c_1, \ldots, c_k$ form an open tuple with respect to the box in question, and analogously, if the cells are row- or column-associated.

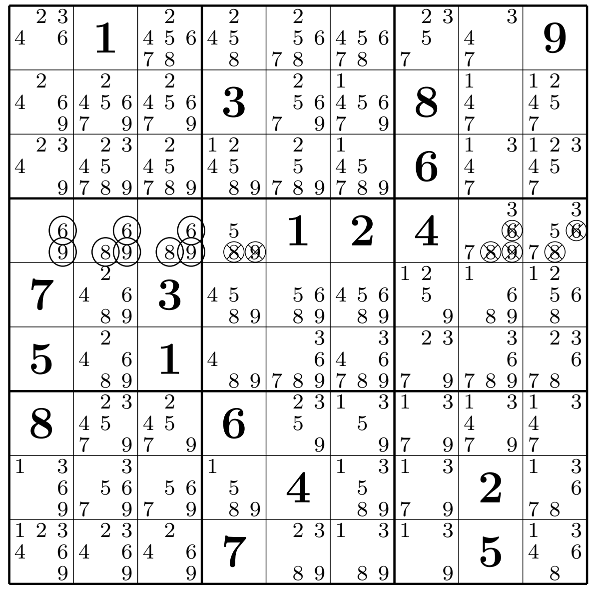
In the sudoku literature, open tuples of cardinality 2 are know as *open* (or *naked*) *pairs*, open tuples of cardinality 3 as *open triples*, etc. Open tuples of cardinality 1 we consequently call *open singletons*.

We again return to the minimal sudoku (example 1.4) with just 17 clues. As shown in section 4, it can be completed with rules F, N, and B. It can also completed with F, N, and T.

Example 5.1 (Minimal sudoku)

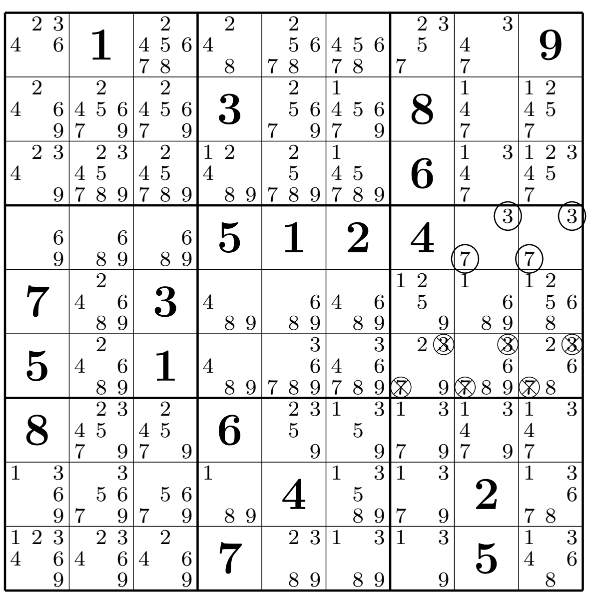


sdk 77

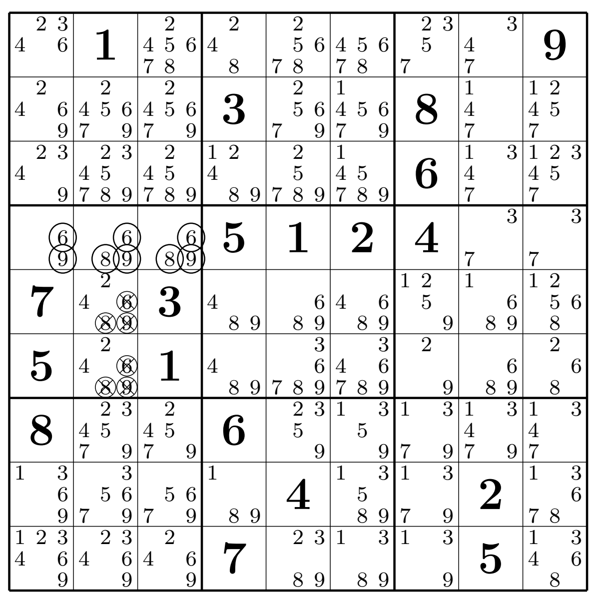


sdk 78

According to rule N_B, digit 1 has to be put into cell $(6, 3)$ of sudoku 77. Thus we get sudoku 78. The candidate table now contains, at the beginning of row 4, the open triple $\{6, 8, 9\}$. Therefore, these candidates can be eliminated in the remaining 6 cells of row 4, and we get sudoku 79

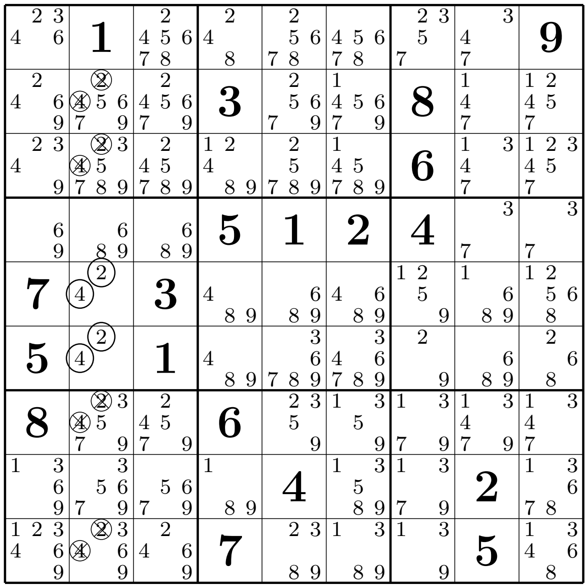


sdk 79

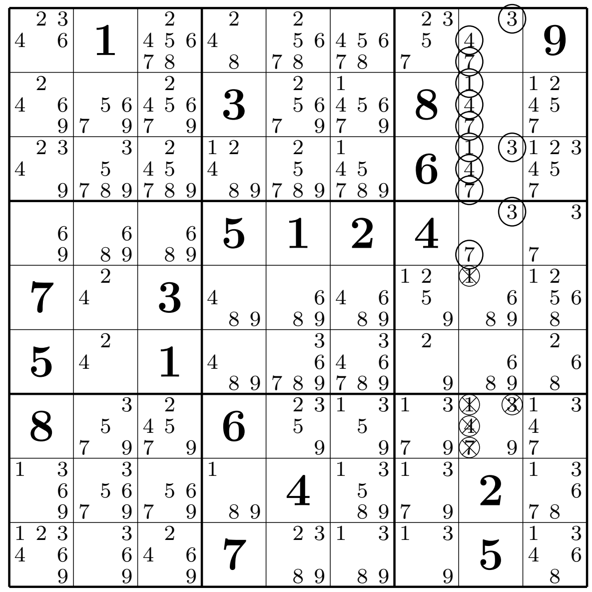


sdk 80

At the end of row 4 of sudoku 79, we now see the open pair $\{3, 7\}$, which allows us to eliminate these candidates in the remaining cells of box $B_{2,3}$. In sudoku 80, we then use the open triple $\{6, 8, 9\}$ at the beginning of row 4 again, but this time with respect not to row 4, but to box $B_{2,1}$, getting sudoku 81.

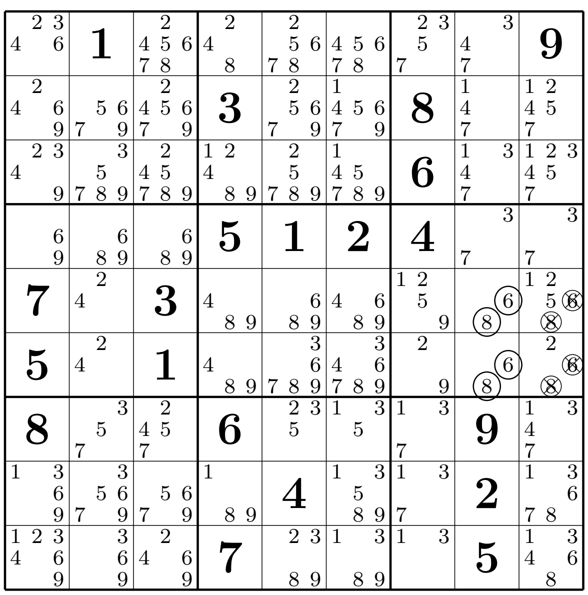


sdk 81

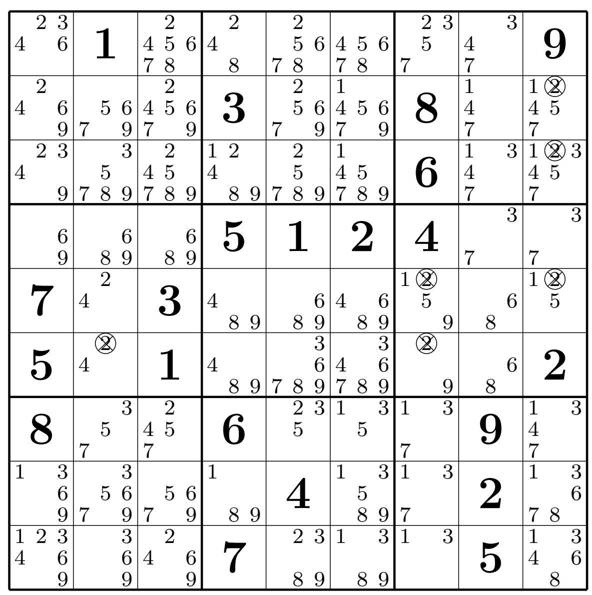


sdk 82

In column 2 of sudoku 81, we now see the open pair $\{4, 2\}$, which allows us to eliminate these candidates in the remaining cells of column 2. In sudoku 82, we then use the open quadruple $\{1, 3, 4, 7\}$ at the beginning of column 8, getting sudoku 83.

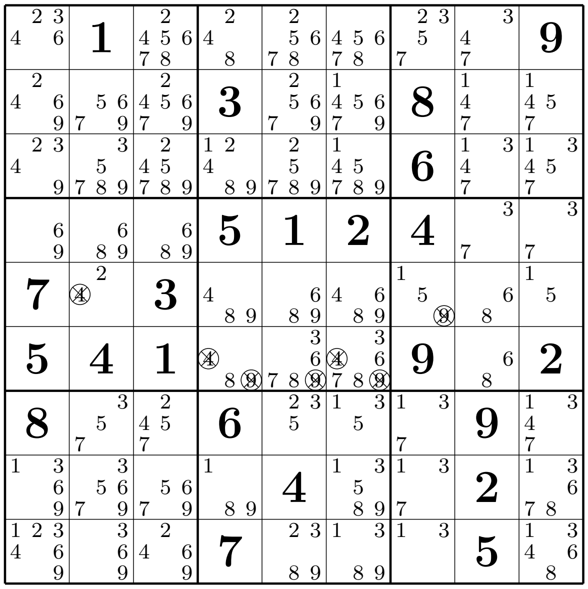


sdk 83

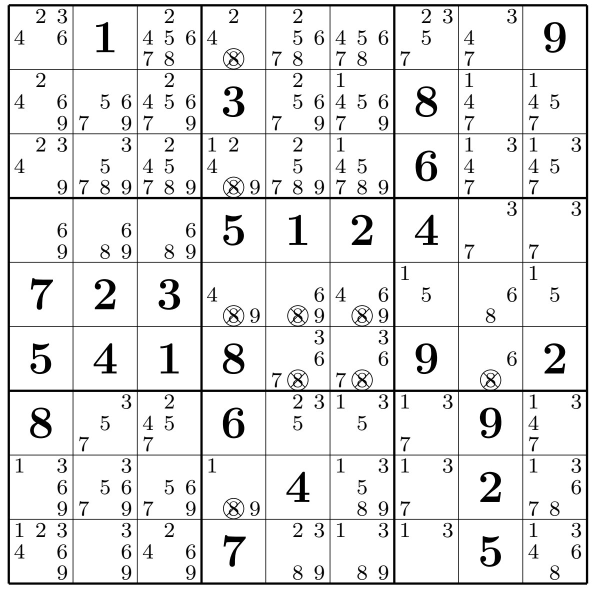


sdk 84

In box $B_{2,3}$ of sudoku 83, we exploit the open pair $\{6, 8\}$ to get sudoku 84. From then on to the end (sudoku 88), we exclusively apply rule F for candidate elimination. In sudoku 84, cells $(6, 2)$ and $(6, 7)$ turn out to have only one candidate left (4 and 9, respectively).

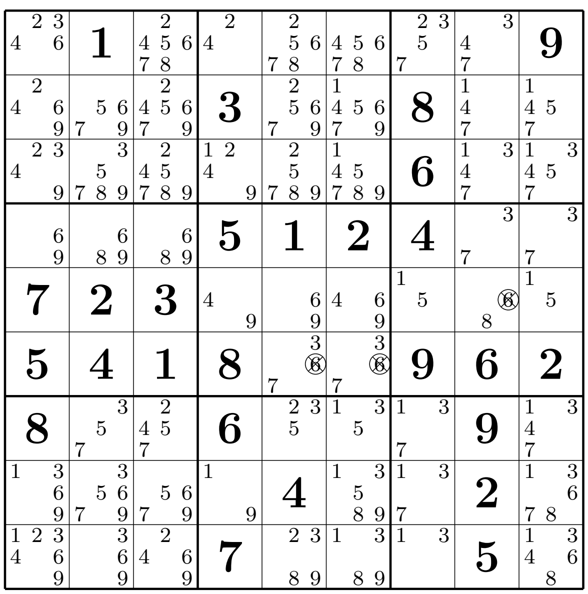


sdk 85



sdk 86

In sudokus 85 and 86, cells $(6, 4)$ and $(6, 8)$ have only one candidate left (8 and 6, respectively).



sdk 87

	1							9
			3			8		
						6		
			5	1	2	4		
7	2	3					8	
5	4	1	8			9	6	2
8			6				9	
				4			2	
			7				5	

sdk 88

In sudoku 87 , finally, rule F can be applied to cell $(5, 8)$, in which the only remaining candidate is 8. Now sudoku 88 is elementary, i.e. can be completed by FN alone.

Remark. The previous sudokus up to and including 87 are not elementary. This might seem to contradict the observation that from sudoku 84 on, we only used the elementary rule F. However, if we start out to complete, for instance, sudoku 87 "from scratch", we loose the previous reductions of the candidate list.

Rule 3 (Tuple reduction T)

Rule T can be split up into three subrules T_B, T_R, and T_C:

T_B *The candidates appearing in an open tuple lying within a common box can be eliminated in the remaining unoccupied cells of the box.*

T_R *The candidates appearing in an open tuple lying within a common row can be eliminated in the remaining unoccupied cells of the row.*

T_C *The candidates appearing in an open tuple lying within a common column can be eliminated in the remaining unoccupied cells of the column.*

5.2 Hidden tuples

Although the process of tuple reduction is defined solely by reference to *open* tuples, it can sometimes be made more convenient by the use of *hidden* tuples.

Definition 10 (Hidden tuples) *Let t be a tuple consisting of* all *the unoccupied cells with respect to some box, or row, or column. Suppose that t contains a subtuple t_1 of k cells such that some k candidates occurring in t are restricted to t_1. Then the remaining $n-k$ cells of t neccessarily form an open tuple containing only the remaining $n-k$ candidates. Therefore, these remaining candidates can be eliminated within t_1. Thus t splits up into two open tuples.*

Whenever a hidden tuple occurs within a set of associated unoccupied cells, the remaining unoccupied cells form an open tuple. Hidden tuples are logically redundant. But it is often easier to detect a hidden pair as, e.g., the open triple, quadriple, quintuple accompaning it. In sudoku 79 of the above example, in box $B_{2,2}$ we have a hidden tuple $\{3, 7\}$ which would lead to the elimination of candidates 4, 6, 8, 9 in cells $(6, 5)$ and $(6, 6)$. The same effect would be produced by exploiting the open quadrupel $\{4, 6, 8, 9\}$ in the remaining unoccupied cells of box $B_{2,2}$.

Whenever some unoccupied cells c_1, ..., c_k are maximal with respect to a box (or a row, or a column), i.e. there are no more unoccupied cells in that box (or row, or column), then iterating rule T has the effect of splitting these cells up into subsets of irreducible open tupels. As a matter of fact, the elementary rules are special cases of tuple reduction, either of *open* (F), or of *hidden* (N_B, N_R, N_C) *singletons.* If the reduction process ends up with only open singletons, the sudoku is completed.

Example 5.2 (Open and hidden tuples)

This example illustrates the interplay between open and hidden tuples. By FN, we get the sudoku below right from the sudoku to the left:

	5	1			6			
		4						9
							7	4
2			8		3			
			2		9			3
9	2							
6						5		
			7			4	8	

sdk 88

7	5	1	9	4	6	2 3	2 3	8
8	6	4	3	2	7	1	5	9
3	9	2	5	8	1	6	7	4
2	1 7	5 6 9	8	1 5 6 7	3	7 9	4	1 5 7
1 5	1 3 7 8	3 5 6 9	4 6	1 5 6 7	4 5	2 7 8 9	1 2 6	1 5 7
4	1 7 8	5 6	2	1 5 6 7	9	7 8	1 6	3
9	2	8	4 6	5 6	4 5	3 7	1 3	1 7
6	4	7	1	3	8	5	9	2
1 5	1 3	3 5	7	9	2	4	8	6

sdk 89

In row 5 of sudoku 89, we find the hidden tuple $\{3, 8, 9\}$ in cells $(5, 2)$, $(5, 3)$, and $(5, 7)$. Therefore, the remaining candidates necessarily form an open tuple in the remaining empty cells, i.e., there is the open tuple $\{1, 2, 4, 5, 6, 7\}$. This, however, is not irreducible. It can be split up into the open quintuple $\{1, 4, 5, 6, 7\}$ in cells $(5, 1)$, $(5, 4)$, $(5, 5)$, $(5, 6)$, $(5, 9)$, and the open singleton $\{2\}$ (also found by N_R) in cell $(5, 8)$. Conversely, $\{2, 3, 8, 9\}$ is a hidden quadrupel in cells $(5, 2)$, $(5, 3)$, $(5, 7)$, and $(5, 8)$ (which is, of course, not irreducible). For completeness, we add that the whole row can be viewed as an open 9-tuple.

Quite independently from the order in which we exploit open and hidden tuples, row 5 is finally split up into the irreducible open tuples $\{2\}$, $\{3, 8, 9\}$, and $\{1, 4, 5, 6, 7\}$. Then completion can be attained by rule F alone.

5.3 Specifying the use of tuples

We may give more detailed information on the application of the T rules by writing, for example, T_4^2. By this, we mean that we take into consideration open tuples of size up to 4 and hidden tuples of size up to 2. If we are just exploiting open pairs, we may write T_2^0 or T_2. Application of hidden tuples up to size 3 can be indicated by T_0^3 or T^3.

5.4 Problems

The six sudokus below can be completed by FN and open pairs (FN+T_2):

	4						1	
9			4		7			5
			1		6			
	7	6				4	3	
				2				
	2	8				5		
			3		9			
7			5		1			6
	8						7	

sdk 90

	9							
	4	1	6	8				5
				7			4	
							7	
	1	8				3	9	
	2							
	5			2				
2	6			5	9	4	8	
							3	

sdk 91

		5				2		
	6		9		4		8	
8								1
	4				8		1	
				6				
	1		4		2		3	
5								3
	9		3		1		7	
		2				9		

sdk 92

9		4						2
		5		6		3		
				2				
		7			4	5	6	
			5		6			
	3	6	8			1		
				5				
		1		7		6		
8						2		1

sdk 93

1			**2**		**3**			**5**
		5		**9**	**8**	**3**		
	5	**2**				**7**	**8**	
	6						**5**	
	9	**3**				**2**	**1**	
		8	**9**	**7**	**4**	**5**		
6			**8**		**2**			**9**

sdk 94

		2				**9**		
			9		**5**			
	3		**2**		**6**		**8**	
	8					**6**	**3**	
				6				
	6	**5**				**8**	**9**	
	5		**6**		**9**		**2**	
			3		**8**			
		7				**4**		

sdk 95

The following four sudokus can be completed by FN and hidden pairs ($FN{+}T^2$). Alternatively, they can be completed by FN and open pairs and triples ($FN{+}T_3$):

		3		**8**		**4**		
9								**7**
8			**7**					**2**
			4		**7**	**6**		
5								**3**
		6	**1**		**3**			
7					**9**			
4								**9**
		1		**6**		**8**		

sdk 96

	2			**4**			**5**	
3						**7**		**8**
	7						**3**	
			3		**5**			
5								**1**
			8		**1**			
	3						**7**	
8		**2**				**1**		**9**
	9			**6**			**8**	

sdk 97

	2		1		4		3	
4				3				8
		6				9		
1								9
	5			2			8	
9								
		7				5		
8				5				3
	1		6		2		9	

sdk 98

		2				7		
		6	5				4	
1	7			8				6
	5				2			
		3		6		2		
			9				7	
3				4			9	1
	9				5	6		
						3		

sdk 99

The following two sudokus can be completed by FN and hidden pairs ($FN{+}T^2$). Alternatively, they can be completed by FN and open pairs, triples, and quadrupels ($FN{+}T_4$):

		3		5		4		
	8		1		4		9	
7								2
	5						6	
3								9
	2						1	
6								8
	1		6				5	
		7		4		6		

sdk 100

	7				6			
4			1			8		
					2		9	
	1				3	7		4
				8				
6		5	7				3	
	3		5					
		9						2
			4				6	

sdk 101

The following two sudokus can be completed by FN, open and hidden pairs (FN+T_2^2). Alternatively, they can be completed by FN and open triples (FN+T_3).

	4		5				1	
7			2					4
		2		9				
9	3							
				6		3		
							2	8
				1		9		
4					9			7
	7				3		6	

sdk 102

	1	8		2				
					9			4
		4		3		6		5
	3							
7				5		9		3
							1	
2		3		4		8		
5			1					
				7		5	4	

sdk 103

The following two sudokus can be completed by FN and hidden triples (FN+T^3). Alternatively, they can be completed by open tupels of length up to 4 (FN+T_4), and up to 5 (FN+T_5), respectively.

		1	6		7	8		
6			3	5				9
5						4		3
		2		4		1		
8		9						6
4				6	3			8
		7	9		2			

sdk 104

	5	1			6			
		4						9
							7	4
2			8		3			
			2		9			3
9	2							
6						5		
			7			4	8	

sdk 105

The last problem is the sudoku of example 5.2.

The following four sudokus require, aside from FN, rule B as well as rule T.

7			9		1			5
	3			6			2	
9				3				6
	7		4		6		3	
8				7				9
	2			4			1	
1			5		9			

sdk 106

7			2		5		8	4
1		6		9			7	
	7		9				4	
				8				
	4				1		3	
	9			6		5		1
5	1		8					3

sdk 107

		8			6		2	
		6	9					4
1	4	9		7				
	2							1
		5				3		
4							8	
				6			3	5
3					5	4		
	9		1			8		

sdk 108

		1			7	8		
6			3	5				9
5						4		3
		2		4		1		
8		9						6
4				6	3			8
		7	9		2	5		

sdk 109

6 X-Chains (One-Candidate Chains)

6.1 Cell chains

Definition 11 (Cell Chain) *By a* cell chain *we understand a sequence of cells such that every two consecutive cells are associated.*

A cell chain is *cyclic*, if some cell appears more than once in it, *non-cyclic* else.

Although cell chains can be investigated as a topic in itself, we always presuppose the presence of a candidate table. Therefore, every cell chain automatically corresponds to a unique *sequence of candidate sets*.

6.2 Strong and weak edges

Any two associated cells determine an *edge*. If we feel more comfortable with a precise definition, an *edge* is just a pair of associated cells. Although we might think of an ordered pair (a directed edge), this is not necessary for our purpose. So we can think of an edge as of an *unordered pair*, or 2-element set, of associated cells.

Definition 12 *An edge is said to be*

(i) weak *with respect to a candidate, if there is a third cell containing the candidate which is associated with both end cells of the edge,*

(ii) strong *otherwise.*

6.3 X_1-Chains

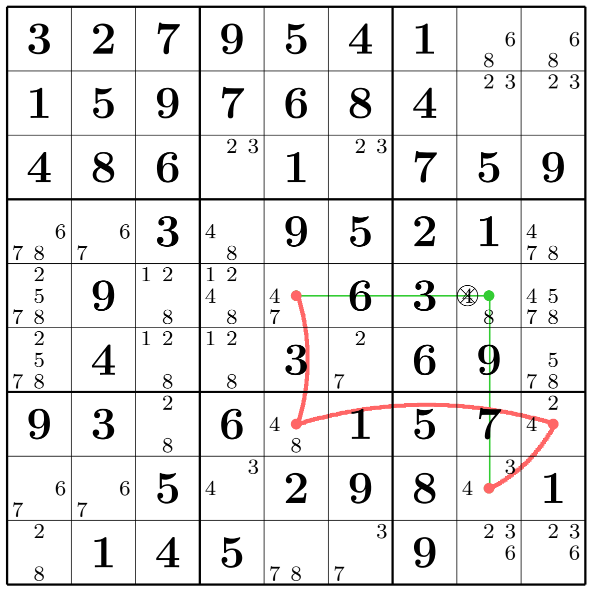
Definition 13 *An* x_1-chain *(with respect to some candidate c) is a chain consisting of an odd number of strong edges (strong with respect to c).*

In any completion, the candidate will be assigned to exactly one of the end cells of an x_1-chain. Therefore, we have the following rule:

Rule 4 (X_1) *If, with respect to some candidate, a sequence of cells form an x_1-chain, then the candidate can be eliminated from any cell associated with both end cells of the chain.*

Remark. In Fowler[2], the edges associating the additional cell with both ends of the chain are included to form a cycle. Distinction is made between x_1-cycles and x_3-cycles. In x_1-cycles, the links are of the same kind, e.g. both with respect to a box, or a row, or a column. In x_3-cycles, they are of different kinds, e.g. one with respect to a row, the second with respect to a column.

In the example below, we find an x_1-chain with respect to candidate 4. (There is even a second x_1-chain, which we will, however, leave aside.)

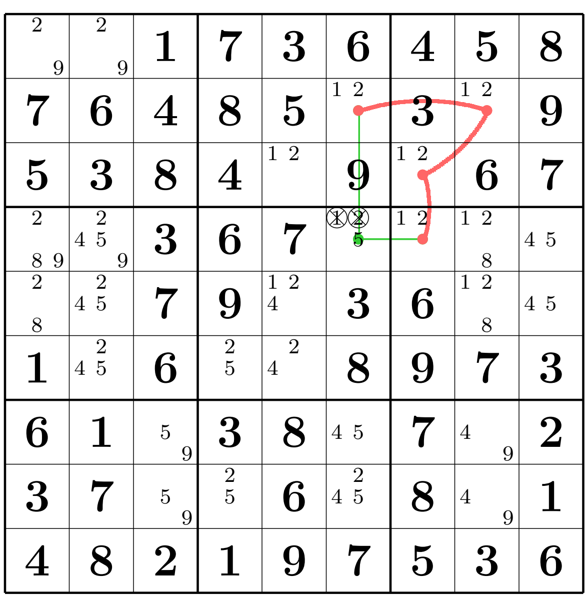


sdk 110

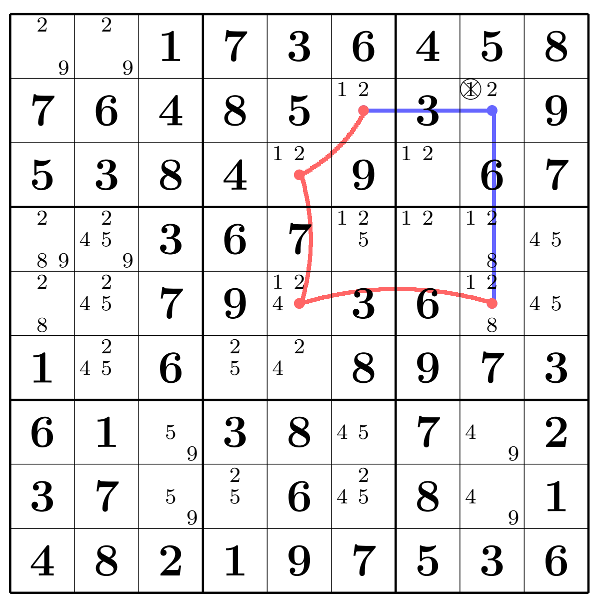
Therefore, candidate 4 can be eliminated from cell (5,8). Completion then only requires FN.

Example 6.1 (Remote pairs)

The technique of *remote pairs* can be replaced by twice applying rule 4 (X_1). In order to explain this, we take the example from SadMan[6]:



sdk 111



sdk 112

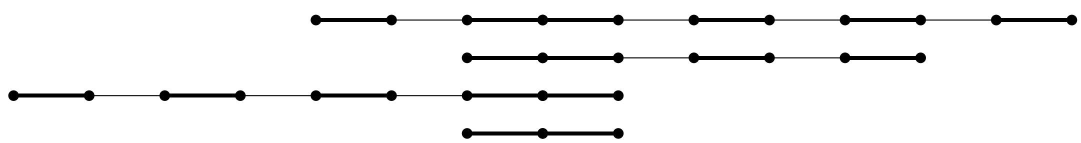
The chain on the left connects 4 cells with candidate set $\{1, 2\}$. The number of edges being odd, we have so-called remote pairs, which allows us to eliminate both candidates from cell $(4, 6)$.

However, there is no need for a special rule. The chain in question is simply an x_1-chain with respect to candidate 1 as well as to candidate 2. Therefore, both can be eliminated from cell $(4,6)$ by rule 4 (X_1).
The chain on the right is, however, only an x_1-chain with respect to candidate 1, because only one of the three edges is strong with respect to 2.
Now 5 has to be assigned to cell (4,6), and 2 to cell (2,8). Completion then only requires FN.

6.4 X_2-Chains

Definition 14 *An* x_2-chain *with respect to some candidate is a chain consisting of a subchain of an* even *number of strong edges and an arbitrary number of subchains each consisting of an* odd *number of strong edges with respect to the given candidate, whereby any two of these subchains are connected by exactly one weak edge.*

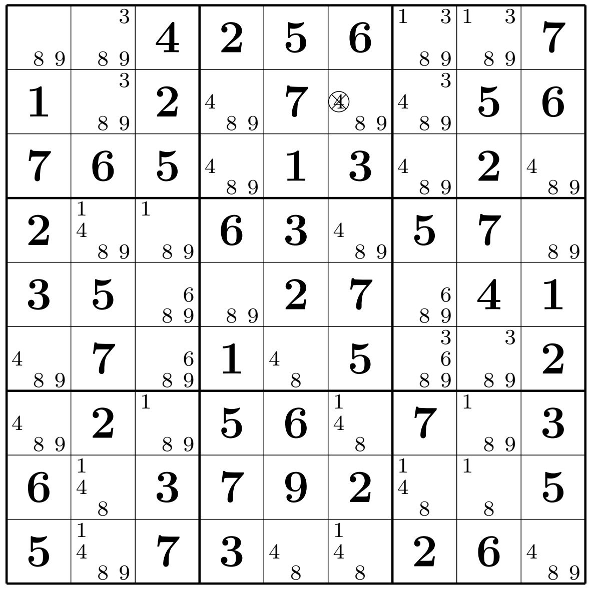
The following figur shows schematically some possibilities for x_2-chains.



Thick lines mean strong, thin lines mean weak edges. The subchain of an even number of strong edges is represented by two strong edges. Every isolated strong edge can be replaced by a subchain of an odd number of strong edges, but every weak edge remains exactly one weak edge.

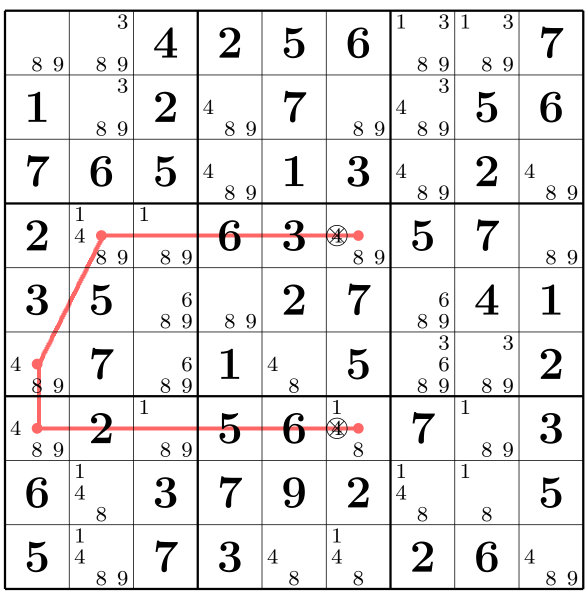
Rule 5 (X_2) *If the end cells of an x_2-chain are associated, then in both end cells of the subchain formed by the even number of strong edges the candidate can be eliminated.*

For assume that the candidate be assigned to one of the end cells of the subchain with the even number of strong edges. Then it would have to be assigned to the other end cell of this subchain as well as to both end cells of the complete chain. But this is a contradiction, as these end cells are supposed to be associated.
The following example is taken from SadMan[7]. Candidate 4 in cell (2,6) can be removed by rule $C \succ B$. (In column 4, candidate 4 is restricted to box $B_{1,2}$.)

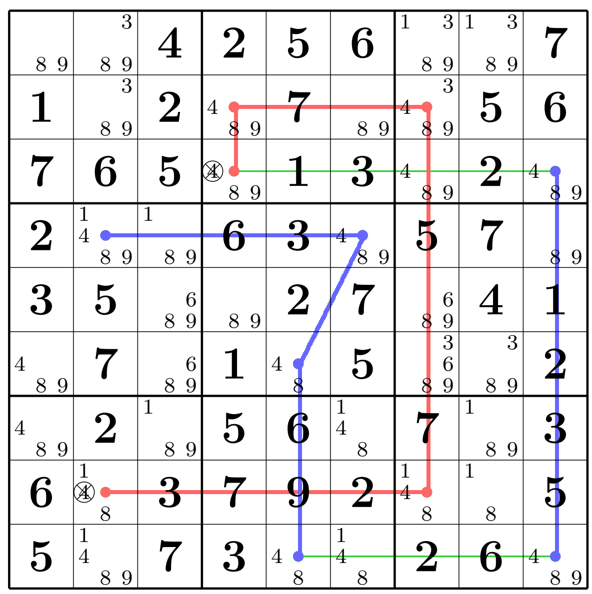


sdk 113

Then for candidate 4 exist no less than 5 x_2-chains. Here are two of them:



sdk 114



sdk 115

The x_2-chain in the sudoku on the left simply consists of 4 strong edges, and the end cells are column-associated. Therefore by rule X_2, candidate 4 can be eliminated from cells $(4,6)$ and $(7,6)$. Now rule N_C says that we have to set cell $(9,6)$ to 4. Completion then only requires FN.

The fancy chain on the right would allows us to eliminate candidate 4 from cells (3,4) and (8,2). Then by rule N_C, we could assign digit 4 to cell(2,4), and by rule N_R, digit 4 to cell (8,7). This alone would not yet, however, leave a sudoku which could be completed by FN alone. We would still have to make use of the chain in the sudoku on the left.

6.5 Problems

The two sudokus below can be completed by using methods FNBT and an x_1-chain:

	5				3			7
6			8	5				
						1		
	7							2
8	9			4			6	3
3							7	
		8						
				3	2			9
9			7	6			5	

sdk 116

			6			5		
	2		3			4	1	
5	6			7				
							3	9
		5				6		
8	9							
				3			9	7
		1			4		6	
		8			5			

sdk 117

The next two sudokus can be completed by using methods FNBT and an x_2-chain:

		8				1		
3	9						2	8
	7		8		9		4	
4				7	2			6
2			5	1	8			9
	2		6		3		5	
9	1						6	4
		7				9		

sdk 118

			1			2		
	8				6		1	
5					9			
	5	9	8		4			3
				9				
7					5	4	2	
			3					4
	7		4				6	
		2			8			

sdk 119

The next two sudokus can be completed by using methods FNBT and an x_1- as well as an x_2-chain (not necessarily in this order):

	4						3	
5				1				4
		1	7			8		
			9		7	1		
	7			6			2	
		6	2		8			
					6	7		
4				5				8
	9						4	

sdk 120

			3			4		
					2			
	9			1		6		5
7							2	
		3		4		7		
	5							6
2		5		9			8	
			6			1		
		7			8			

sdk 121

6.6 Hints to the problems

By FNB and $FNBT_2$, respectively, from the first two sudokus we get:

1 2 4	**5**	1 2 4 9	1 2 4 9	1 2 9	**3**	**6**	**8**	**7**
6	1 2 3	**7**	**8**	**5**	1 9	2 3	2 3 9	**4**
2 4	**8**	2 3 4 9	2 4 6 9	**7**	4 6 9	**1**	2 3 9	**5**
4 5	**7**	4 5 6	**3**	**8**	5 6 9	4 9	**1**	**2**
8	**9**	1 2	1 2	**4**	**7**	**5**	**6**	**3**
3	1 2 4 6	1 2 4 5 6	1 2 5 6 9	1 2 9	1 5 6 9	4 9	**7**	**8**
1 2 5	1 2 3	**8**	1 4 5 9	1 9	1 4 5 9	**7**	2 3	**6**
7	1 6	1 5 6	1 5	**3**	**2**	**8**	**4**	**9**
9	2 3 4	2 3 4	**7**	**6**	**8**	2 3	**5**	**1**

sdk 116

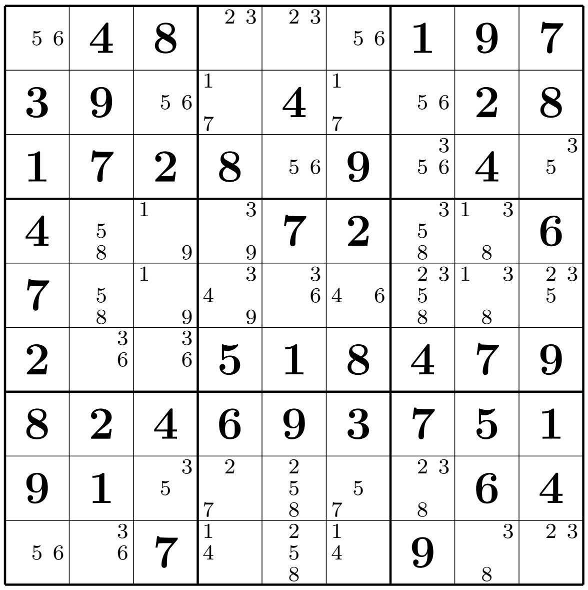
1	**8**	3 4	**6**	2 4	**9**	**5**	**7**	2 3
7	**2**	**9**	**3**	**5**	**8**	**4**	**1**	**6**
5	**6**	3 4	1 2 4	**7**	1 2	**9**	2 8	2 3 8
2 6	**4**	2 6 7	**5**	**8**	1 2 6	1 2 7	**3**	**9**
3	**1**	**5**	2 4 9	2 4 9	**7**	**6**	2 4 8	2 4 8
8	**9**	2 6 7	1 2 4	1 2 4 6	**3**	1 2 7	**5**	1 2 4
4	**5**	2 6	1 2 8	**3**	1 2 6	1 2 8	**9**	**7**
2 9	3 7	**1**	7 8	2 9	**4**	3 8	**6**	**5**
2 6 9	3 7	**8**	1 2 7 9	1 2 6 9	**5**	1 2 3	2 4	1 2 4

sdk 117

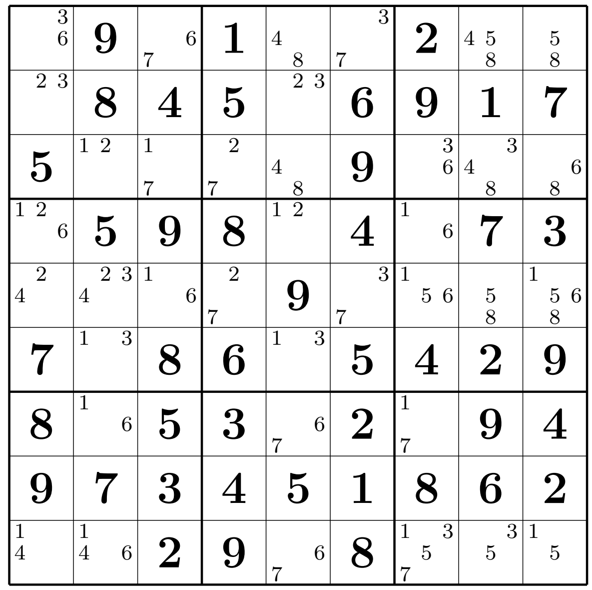
Sudoku 116 contains the chain $((2,7),\ (9,7),\ (7,8),\ (7,2))$, which is an x_1-chain with respect to candidate 3. Therefore by rule 4, candidate 3 can be eliminated from cell $(2,2)$. As an immediate consequence, cell $(3,3)$ has to be put to 3. Then completion is possible by FNB.

Sudoku 117 contains the chain $((4,1),\ (9,1),\ (9,5),\ (7,6))$, which is an x_1-chain with respect to candidate 6. Therefore, candidate 6 can be eliminated from cell $(4,6)$. As an immediate consequence, cells $(6,5)$ and $(7,6)$ have to be set to 6 (rules N_B and N_C). Then completion is possible by FN alone.

The next two sudokus can be extended by $FNBT_2^2$ and $FNBT_4$, respectively, to



sdk 118



sdk 119

Sudoku 118 contains the chain ((3, 5), (1, 6), (1, 1), (9, 1), (9, 5)), which is an x_2-chain with respect to candidate 5. It is a "simple" x_2-chain in the sence that it contains no weak edges, but just consists of 4 strong edges with respect to 5. Therefore by rule 5, candidate 5 can be eliminated from cells (3, 5) and (9, 5). As an immediate consequence, cell (3, 5) can be put to 6. Then completion is possible by F alone.

Sudoku 119 contains the chain ((7, 7), (7, 2), (3, 2), (3, 3), (5, 3), (5, 9), (9, 9)), which is an x_2-chain with respect to candidate 1. This chain is not "simple". The core strong-edge part is ((3, 2), (3, 3), (5, 3)). At both ends, a weak edge is added, and then one single additional strong edge. Therefore, candidate 1 can be eliminated from cells (3, 2) and (5, 3), which are the ends of the core strong-edge part. As an immediate consequence, cells (3, 2) and (5, 3) can be set to 2 and 6, respectively. Then completion is possible by F alone.

The next two sudokus can be extended by $FNBT_2$ and $FNBT_3$, respectively, to

7	**4**	289	568	289	259	269	**3**	**1**
5	268	289	368	**1**	239	269	**7**	**4**
2369	236	**1**	**7**	29	**4**	**8**	569	259
23	235	2345	**9**	34	**7**	**1**	**8**	**6**
389	**7**	3489	15	**6**	15	349	**2**	39
139	13	**6**	**2**	34	**8**	3459	59	**7**
1238	12358	2358	**4**	289	**6**	**7**	159	2359
4	1236	**7**	13	**5**	1239	2369	169	**8**
12368	**9**	2358	138	**7**	123	2356	**4**	235

sdk 120

1568	178	168	**3**	5678	569	**4**	179	**2**
13456	1347	146	4579	567	**2**	89	1379	18
348	**9**	**2**	478	**1**	47	**6**	37	**5**
7	48	4689	**1**	68	69	**5**	**2**	**3**
689	**2**	**3**	589	**4**	569	**7**	19	18
189	**5**	189	**2**	37	37	89	**4**	**6**
2	**6**	**5**	47	**9**	**1**	**3**	**8**	47
3489	348	489	**6**	**2**	347	**1**	**5**	479
1349	134	**7**	45	35	**8**	**2**	**6**	49

sdk 121

Sudoku 120 contains the chain $((3,1), (9,1), (8,2), (8,8), (3,8))$, which is an x_2-chain with respect to candidate 6. The core even-edged part is $((3,1), (9,1), (8,2))$, and therefore, candidate 6 can be eliminated from cells $(3,1)$ and $(8,2)$. Therefore, cell $(9,1)$ can be set to 6 by rule N_B. As now candidate 8 disappears from this cell, the edge $((9,3), (9,4))$, which was a weak edge for candidate 8, now becomes a strong edge, and therefore $((1,5), (7,5), (9,4), (9,3))$ turns into an x_1-chain for candidate 8. This leads to the elimination of candidate 8 from cell $(1,3)$. Completion can then be achieved by FNB.

Sudoku 121 contains the chain $((1,8), (1,6), (2,4), (5,4))$, which is an x_1-chain with respect to candidate 9. Therefore, candidate 9 can be eliminated from cell $(5,8)$, which implies that this cell can be set to 1, and cell $(5,9)$ can be set to 8. Now by rule N_B, cells $(4,5)$ and $(3,4)$ can be both set to 8. Edge $((2,4), (3,6))$ becomes a strong edge with respect to candidate 4. By rule F, cell $(4,2)$ can be set to 4, which makes $((2,3), (8,3))$ a strong edge with respect to candidate 4. The chain $((2,4), (3,6), (8,6), (8,3), (2,3))$ turns into an x_2-chain with the core strong-edged part $((2,4), (3,6), (8,6))$. By rule X_2, candidate 4 can be eliminated from cells $(2,4)$ and $(8,6)$. Then completion only requires rule F.

7 Pair Chains (Y-Chains)

In the presence of a candidate table, to every cell chain corresponds a sequence of candidate sets. Because in a chain, any two consecutive cells are associated, every possible assignment to a chain is a sequence of candidates such that any two consecutive members are distinct. This leads us to define:

Definition 15 (Homogeneous, strictly inhomogeneous sequences) *(i) We call a sequence* homogeneous*, if all elements are equal.*

(ii) We call a sequence strictly inhomogeneous*, if no two consecutive elements are equal.*

By this definition, the terms *homogeneous* and *strictly inhomogeneous* can be applied to candidate sequences as well as to pair sequences. Note that $(\{1, 2\}, \{1, 3\}, \{1, 2\})$ is strictly inhomogeneous, although the first element coincides with the last. We now turn our attention to the case where all candidate sets are (unordered) pairs, i.e. consist of exactly two distinct candidates.

7.1 Y-sequences and y-chains

Definition 16 (Y-sequence, y-chain) *A* y-sequence with respect to c_0 *is a sequence of pairs* $(\pi_1, \ldots, \pi_n)$ *such that, for some strictly inhomogeneous sequence* $(c_0, c_1, \ldots, c_{n-1}, c_0)$, $\pi_i = \{c_{i-1}, c_i\}$ *for* $i = 1, \ldots, n-1$, *and* $\pi_n = \{c_{n-1}, c_0\}$, *i.e. a pair sequence that can be written in the form*

$$\Pi = (\{c_0, c_1\}, \{c_1, c_2\}, \ldots, \{c_{n-2}, c_{n-1}\}, \{c_{n-1}, c_0\}).$$

A y-chain with respect to some candidate c_0 *is a chain such that the corresponding sequence of candidate sets is a y-sequence with respect to* c_0.

The wording "can be written in the form" reminds of the fact that candidate sets are *unordered* pairs (2-element sets). Therefore, $\{a, b\} = \{b, a\}$ for any a, b. For *ordered* pairs, however, $(a, b) \neq (b, a)$ if $a \neq b$. Note that braces are used for (unordered) pairs, and parentheses for ordered pairs. If candidate c_0 is *not* assigned to the first cell, then the assignments necessarily are

$$c_1, c_2, \ldots, c_{n-1}, c_0,$$

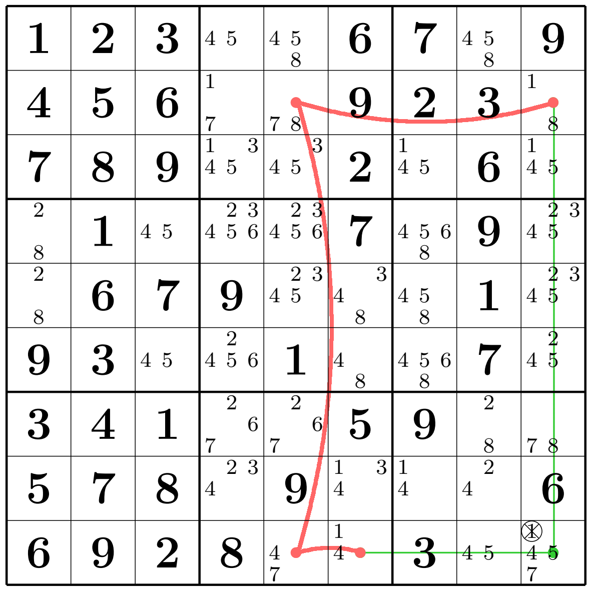
i.e. c_0 is assigned to the last cell. As an immediate consequence, we obtain:

Rule 6 (Y) *If, with respect to some candidate, a sequence of cells forms a y-chain, then in any cell associated to both ends of the chain, the candidate can be eliminated.*

Sometimes, e.g. in GoldenChain[5], y-chains are named *golden chains*. The simplest y-chain is usually called *xy-wing*. It consists of exactly two edges.

Example 7.1

A y-chain occurs, for example, in "puzzle y - 1" of FOWLER[2]:



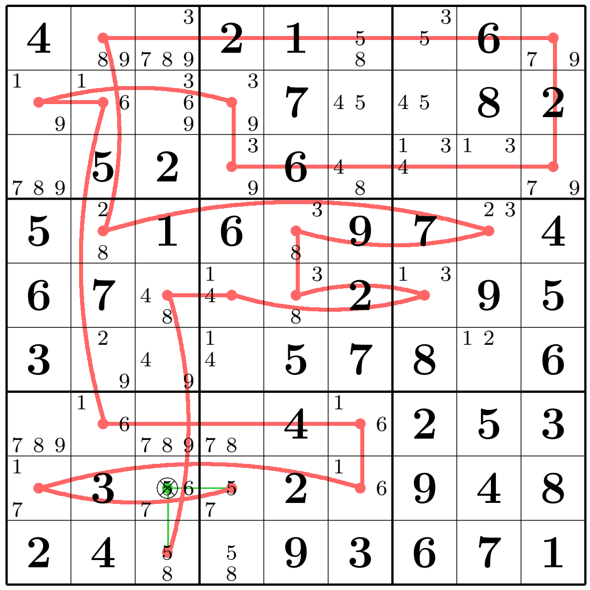
sdk 122

The marked cell chain is $((2,9),\ (2,5),\ (9,5),\ (9,6))$. It is a y-chain with respect to candidate 1, as the corresponding pair sequence can be written as $(\{1,8\},\{8,7\},\{7,4\},\{4,1\})$. Hence by rule 6, candidate 1 can be eliminated from cell (9,9), which is associated to both ends of the chain. Then by rules N_B and N_R, cells $(8,7)$ and $(9,6)$ can both be set to 1. The resulting sudoku is, however, not yet elementary.

Yet there is also the y-chain $((2,4),\ (2,9),\ (7,9))$, to which corresponds the y-sequence $(\{7,1\},\ \{1,8\},\ \{8,7\})$. It allows us to eliminate candidate 7 from cell $(7,4)$. Then by rule N_C, cell $(2,4)$ has to be set to 7. Now the resulting sudoku is elementary.

Example 7.2

A rather intricate example of a y-chain is presented in FOWLER[2] ("puzzle y - 2"):



sdk 123

It starts at cell (8,4) and ends at cell (9,3). It comprises 19 edges, and therefore 20 cells. The corresponding sequence of candidate pairs can be written as

$$(\{5,7\},\ \{7,1\},\ \{1,6\},\ \{6,1\},\ \{1,6\},\ \{6,1\},\ \{1,9\},\ \{9,3\},\ \{3,9\},\ \{9,7\},\ \{7,9\}, \{9,8\},\ \{8,2\},\ \{2,3\},\ \{3,8\},\ \{8,3\},\ \{3,1\},\ \{1,4\},\ \{4,8\},\ \{8,5\}). \quad (1)$$

It is a y-sequence with respect to candidate 5. Hence by rule (6), this candidate can be eliminated from cell $(8,3)$.

As a matter of fact, the above sudoku can be completed without recourse to this chain, which is very hard to spot. Indeed, the cell chain $((1,6),(1,7),(5,7),(5,4),(5,3))$ turns out to be a y-chain for candidate 8, which therefore can be eliminated from cell $(1,3)$. The corresponding pair sequence can be written as $(\{8,5\},\ \{5,3\},\ \{3,1\},\ \{1,4\},\ \{4,8\})$. Likewise, $((3,9),(3,4),(2,4),(2,1),(8,1))$ is a y-chain for candidate 7, which can be eliminated from cell $(3,1)$. Then the solution can be completed by elementary methods alone.

7.2 Domino chains

With the following definition, we slightly generalize the notion of a y-chain:

Definition 17 (Domino sequence, spine; domino chain) *A* domino sequence of type (c_0, c_n) *is a sequence of pairs* $(\pi_1,\ \ldots,\ \pi_n)$ *such that, for some strictly inhomogeneous sequence* $(c_0,\ c_1,\ \ldots,\ c_{n-1},\ c_n)$, $\pi_i = \{c_{i-1}, c_i\}$ *for* $i = 1,\ \ldots, n$, *i.e. a pair sequence that can be written in the form*

$$\Pi = (\{c_0, c_1\}, \{c_1, c_2\}, \ldots, \{c_{n-2}, c_{n-1}\}, \{c_{n-1}, c_n\}).$$

We call the sequence $(c_0, c_1, \ldots, c_{n-1}, c_n)$ *a* spine *of the domino sequence. Note that a domino sequence may have more than one spine (up to 2).*
A domino chain of type (c_0, c_n) *is a chain such that the corresponding sequence of candidate sets is a domino sequence of type* (c_0, c_n).

If $c_0 = c_n$, then the domino chain becomes a y-chain.

Lemma 7.1 (Domino lemma) *If in a domino chain of type* (c_0, c_n)*, the first cell is* not *assigned candidat* c_0*, then the last cell is assigned* c_n.

Proof: If the first cell is not assigned candidate c_0, the assignments necessarily are $c_1, c_2, \ldots, c_{n-1}, c_n$. Q.E.D.

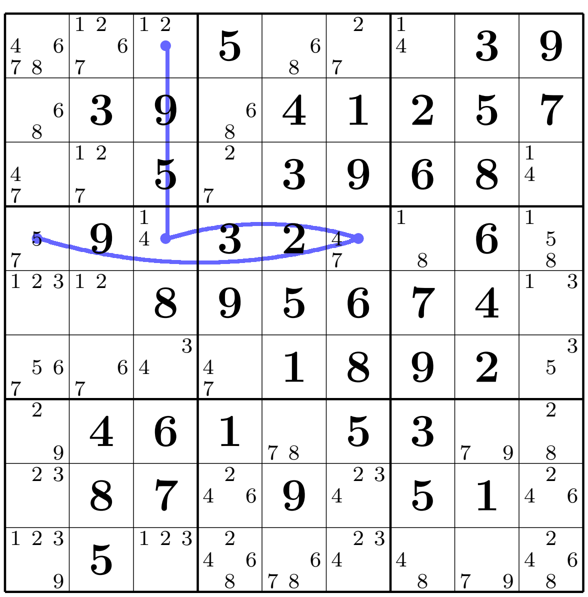
The following lemma is a direct consequence of definition 17:

Lemma 7.2 (Subsequence lemma) *Any subsequence of a domino chain is itself a domino chain.*

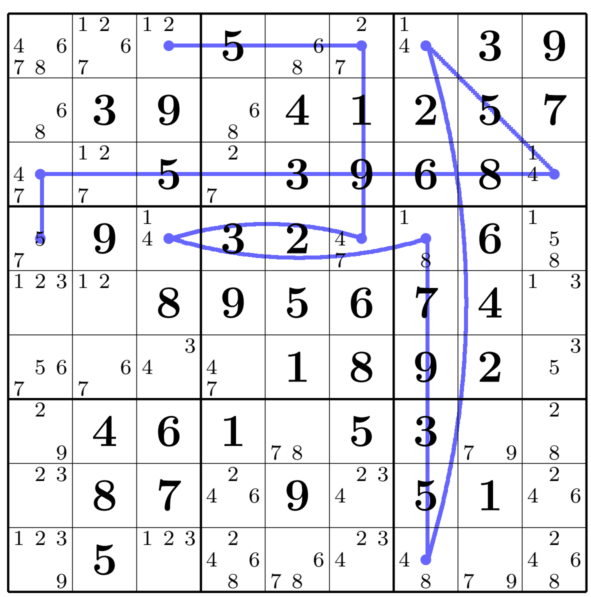
While domino chains do not directly lead to the elimination of candidates, they can help to do so indirectly, as the following example will show.

Example 7.3 (Forcing chains)

In in Palmsudoku[4], *forcing chains* are used to eliminate candidate 7 in cell (4,1):



sdk 124



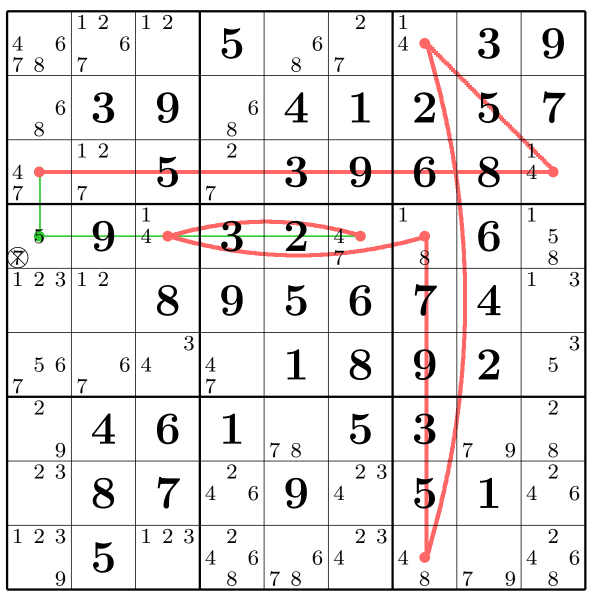
sdk 125

The pictures show two different chains, both beginning at cell (1,3) and ending at cell (4,1). To the path on the left corresponds the pair sequence $(\{2,1\},\{1,4\},\{4,7\},\{7,5\})$, the chain therefore is a domino chain of type $(2,5)$. Thus by lemma 7.1, if 2 is *not* assigned to cell (1,3), then 5 is assigned to cell (4,1). The path in the right picture belongs to a domino chain of type (1,5), as the corresponding pair sequence is

$$(\{1,2\},\{2,7\},\{7,4\},\{4,1\},\{1,8\},\{8,4\},\{4,1\},\{1,4\},\{4,7\},\{7,5\}).$$

Again by lemma 7.1, if 1 is *not* assigned to cell (1,3), then to cell (4,1) necessarily is assigned 5. As a consequence, to cell (4,1) will in both cases be assigned candidate 5, so candidate 7 can be eliminated.

The problem can be completed by use of a single y-chain for candidate 7. It is obtained from the long chain on the right by omission of the first two and the last edge, and is a y-chain with respect to candidate 7. It starts in cell (4,6) and ends in cell (3,1). Therefore by the application of this y-chain, candidate 7 can be eliminated from cell (4,1):



sdk 126

7.3 Problems

The following 10 sudokus can all be completed by $FNBT$ and rule Y. In the first two, one single application of rule Y suffices. In the last two, iteration of $FNBT$ and rule Y is required. Especially the last sudoku contains a plethora of y-chains.

The following two sudokus can be completed by $FNBT$ and a unique application of Y:

7			4	6	5			
	1							
	3			2			9	
	9	6			1		2	
3								1
	2		8			3	5	
	5			1			3	
							6	
			2	3	9		8	7

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6	7							1
		1				4		8
	8				2		9	
		7		1				
			4	6	7			
				5		2		
	3		8				6	
4		6				5		
7								9

sdk 128

In the next two sudokus, FNB leads to sudokus each with 3 y-chains. In each case, application of rules Y and F lead to completion.

		2				9		
			9		5			
	3		2		6		8	
	8	4					3	
				6				
	6	5				8	9	
	5		6		9		2	
			3		8			
		7				4		

sdk 129

	1		8		9	5		
5				4				9
							8	
		3	7	6		1		
	9						5	
		7		9	1	2		
	7						6	
6				8				2
		4	6		7		3	

sdk 130

In the next two sudokus, FNB leads to sudokus with 2 and 3 y-chains, respectively. Rule Y has to be applied to all of them. Completion can then be reached by FN.

			8		3		7	
					1			9
				9				6
4							1	5
		1		3		2		
6	7							3
5				6				
2			9					
	1	3	7		2			

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	2	4			6		7	
						8		
			1	9				6
4				7	8	3	5	
	5	1	6	3				4
2				4	1	7		
		3						
	4		5			9	6	

sdk 132

In the sudoku below left, FNT_3 leads to 4 y-chaines. Application of rule Y to one of them suffices to produce a sudoku which can be completed by FN. (Three of the four y-chaines have this agreeable effect.) The next sudoku requires, besides $FNBT_3$, an iterated application of rule Y.

		6		4				
1				9				
	9	3	1			2		
	1			6				
	7	4	3		2	6	8	
				7			4	
		8			9	1	7	
				8	6			3
				3		4		

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		7		6		9		
	2		7		1		8	
								6
	4						6	
2				8				1
	5						2	
9								7
	7		8		4		9	
		3		9		5		

sdk 134

The following two sudokus require iterated applications of $FNBT$ and rule Y. The second is a real test for finding an almost endless succession of y-chains.

		8	5	6	4			
					2		4	9
	5						2	6
	9	7						
3								4
						2	9	
5	8					4	7	
1	3		7					
			4	1	5	3		

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					4			
	6		8			1	7	9
	9			6				4
4				8		7	3	
	3	1		7				5
2				5	9		8	
3	5	8			7		6	
			6					

sdk 136

7.4 Hints to the problems

sdk 127 By FNB, the sudoku converts into the state $(62, 39)$. By this we mean that 62 cells are occupied and there remain 39 candidates in the empty cells. Then there appears the y-chain $((2,5), (2,9), (6,9))$ for candidate 9. After candidate 9 is eliminated by rule Y, completion can be achieved by rule F alone.

sdk 128 By $FNBT_2$, we reach state $(41, 100)$. Using y-chain $((5,9), (5,2), (6,3))$ for candidate 3, we can achieve completion by N_B and F.

sdk 129 By FNB, we reach state $(57, 52)$, and now there appear 3 y-chaines, all of length 3. Using any one of them, e. g. $((1,1), (1,9), (9,9))$ for candidate 8, we can then achieve completion by F.

sdk 130 By FNB, we reach state $(56, 62)$, and we have 3 y-chaines. Using, for instance, $((4,1), (5,3), (8,3), (8,6))$ for candidate 4, we can achieve completion by F alone.

sdk 131 By $FNBT_2$, we reach state $(51, 74)$. There are now the 2 y-chaines $((3,8), (3,3), (4,3), (5,1))$ and $((5,1), (9,1), (9,5), (8,6), (7,6))$, both for candidate 8. After applying rule Y to both of them, we can reach completion by N and F.

sdk 132 By FNB, we reach state $(56, 55)$. There are now 3 y-chaines: one of three edges for candidate 9, one of 4 edges for candidate 8, and one of 4 edges for candidate 9. We have to apply rule Y to all three of them. Then completion can be achieved by F alone.

sdk 133 By FNT_3, we reach state $(52, 67)$. There are now 4 y-chaines: two of three edges for candidate 5, one of three edges for candidate 9, and one of 4 edges for candidate 5. Three of them lead directly to a state where completion can be achieved by F. Apply, for instance, rule Y to the chain $((1, 2)$, $(6, 2)$, $(6, 7)$, $(5, 9))$.

sdk 134 Here $FNBT$ and rule Y have to be iterated. First, $FNBT_3$ leads to $(37, 116)$. Then there is a 2-edge y-chain for candidate 3, and a 4-edge y-chain for candidate 2. After using both of them, we can get to state $(61, 44)$ by FN. Again we can spot two y-chaines: one for candidate 3 and one for candidate 5, both having 4 edges. After applying rule Y to both of them, we can get completion by rule F.

sdk 135 This sudoku can be completed by almost endlessly iterating $FNBT_2$ and Y.

sdk 136 Completing this sudoku without guessing requires extrem tenacity. We present a possibility:

Start		$(27, 196)$
$FNBT_4^2$		$(35, 124)$
$Y(6)$	$((4, 9)$, $(8, 9)$, $(9, 9)$, $(9, 5$, $(2, 5)$, $(2, 6)$, $(6, 6))$	$(35, 123)$
NBT_2		$(37, 110)$
$Y(1)$	$((1, 5)$, $(5, 5)$, $(8, 5)$, $(7, 4)$, $(7, 7)$, $(7, 2))$	$(37, 109)$
BT_2		$(37, 107)$
$Y(8)$	$((3, 7)$, $(1, 9)$, $(1, 2)$, $(5, 2))$	$(37, 106)$
$Y(8)$	$((1, 2)$, $(5, 2)$, $(5, 9)$, $(8, 9)$, $(9, 9)$, $(1, 9))$	$(37, 105)$
$Y(1)$	$((1, 1)$, $(1, 2)$, $(1, 9)$, $(9, 9)$, $(9, 5)$, $(7, 4))$	$(37, 104)$
$Y(8)$	$((5, 2)$, $(1, 2)$, $(1, 9)$, $(9, 9)$, $(8, 9)$, $(5, 9))$	$(37, 103)$
$Y(7)$	$((1, 1)$, $(1, 5)$, $(5, 5)$, $(8, 5)$, $(8, 9)$, $(9, 9)$, $(1, 9)$, $(1, 2))$	$(37, 101)$
N		$(38, 98)$
$Y(9)$	$((1, 4)$, $(3, 6)$, $(4, 6)$, $(4, 3))$	$(38, 97)$
FN		$(42, 85)$
$Y(3)$	$((1, 9)$, $(1, 2)$, $(5, 2)$, $(6, 1)$, $(6, 6)$, $(2, 6)$, $(2, 5)$, $(9, 5))$	$(43, 83)$
F		$(81, 0)$

In the protocol of sudoku 136, the ordered pair at the end of the line indicates the number of definitely set digits and the total number of remaining candidates. Among the possible y-chaines, we always select the first among the shortest chaines which has an effect, i.e. leads to elimination of at least one candidate. Thereby, cells are ordered in reading order. First come all cells of the top line of the sudoku, then those of the second, and so on. Chaines of equal length are enumerated according to their first cells, and in each chain, the first cell preceeds the end cell in the order of cells.

As there are often very many y-chaines from which we can choose, we have a huge variety of ways to complete a sudoku like the above one. However, the Y rule is necessary. Without it, sudoku 136 cannot be completed by constraint propagation alone. In cases like this, and the more so in even more complicated cases, it may well be reasonable to include some trial and error.

8 W-Patterns (Swordfish, X-Wing)

For any given digit, the candidates in n given rows occupy at least n columns. Otherwise, some column would necessarily have to contain this digit more than once. If the candidates occupy not more columns as rows, they are said to form a *swordfish*. Then in the solution, the digit will, in these rows, occupy exactly these columns (in whatever order). Therefore in these columns, the digit is not possible outside the given rows.

A swordfish in which, in each of the k mentioned rows, the candidates occupy *all* of the k columns is called X-Wing.

Rule 7 (Swordfisch (W)) *If in k rows, the candidates of some given digit occupy just k distinct columns, in these columns the candidates outside the given rows can be eliminated.*

These rules and definitions remain true, of course, if "row" and "column" are interchanged.

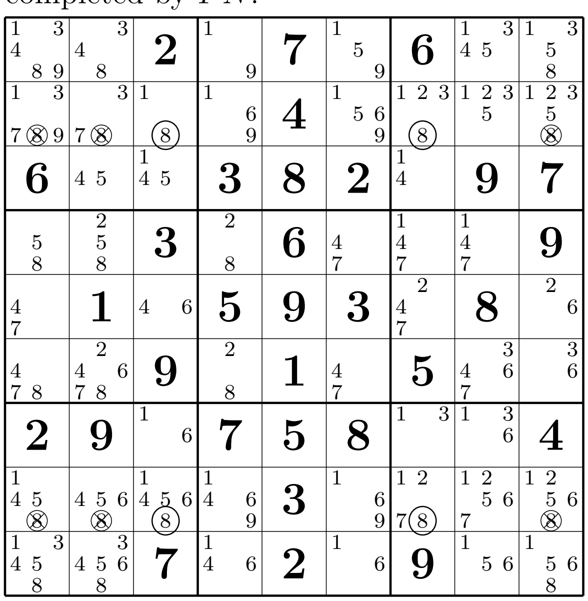
Example 8.1 (X-Wing)

If we apply $FNBT_3$ to the sudoku to the right, we get a sudoku with two x-wings with respect to columns. They are illustrated in the sudokus below, together with the effects.

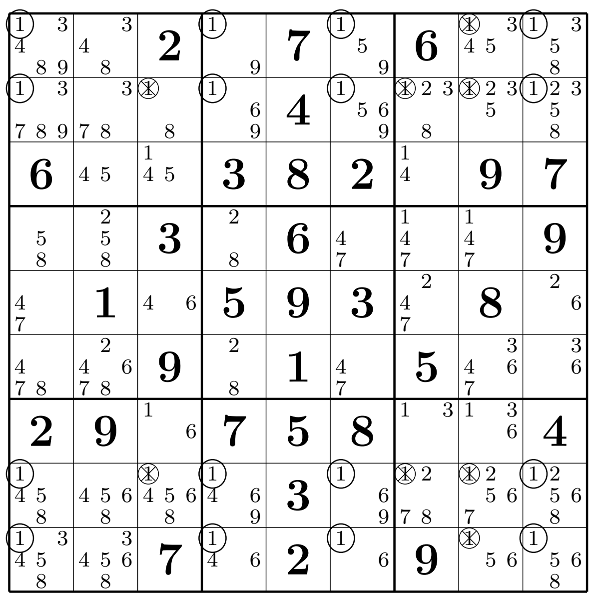
To the left, we see that in columns 3 and 7, candidate 8 is restricted to rows 2 and 8. Therefore, in these rows, candidate 8 can be eliminated outside of columns 3 and 7. To the right, we see that in columns 1, 4, 6, and 9, candidate 1 is restricted to rows 1, 2, 8, and 9. Therefore, in these rows, candidate 1 can be eliminated outside of columns 1, 4, 6, and 9. The sudoku can then be completed by FN.

		2				**6**		
				4				
6			**3**		**2**			**7**
		3		**6**				
	1		**5**	**9**	**3**		**8**	
		9		**1**		**5**		
2			**7**		**8**			**4**
				3				
		7				**9**		

sdk 136



sdk 137



sdk 138

8.1 Problems

The following six sudokus can be completed by $FNBT$ and a single application (in one case two applications) of rule W. In fact, in all cases W just means "x-wing". In some two rows, some candidate is restricted to just two columns.

	1				3	4		
		7	8					1
2				6			9	
8							7	
		9		4		6		
	5							4
	6			5				7
5					1	3		
		3					8	

sdk 139

		8	2		7	1		
			8		3			
1								4
3	2						4	6
9	4						5	3
8								1
			7		6			
		7			2	5		

sdk 140

Both sudokus could also be completed by $FNBT$ and Y. Furthermore, the first could also be completed by $FNBT$ and X.

		7		5			9	
6				8			3	
			7					6
						6		
3	9			1			5	2
		4						
4					3			
	1			7			8	4
	3			9		1		

sdk 141

	8		6		1		7	
9								1
				4				
3				5				4
		4	7		3	2		
1				9				5
				6				
8								2
	6		3		2		8	

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The first sudoku could also be completed by $FNBT$ and X, the second by$FNBT$ and Y.

3			**4**		**5**			**7**
		2				**6**		
5				**1**				**9**
			6		**8**			
9				**7**				**4**
		3				**2**		
4			**9**		**7**			**5**

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	4						**5**	
2						**9**		**6**
	3				**8**			
	2	**7**	**4**		**3**			
				9				
			8		**2**	**4**	**3**	
			7				**9**	
7		**9**	**6**					**5**
	1						**4**	

sdk 144

The first sudoku could also be completed with $FNBT$ and both of X and Yinstead of W, the second by $FNBT$ and X as well as by by $FNBT$ and Y.

8.2 Hints to the problems

sdk 139 By FNB, the sudoku converts into the state $(45, 92)$ (45 digits definitely set, 92 candidates left). Then in rows 5 and 8, candidate 7 is restricted to columns 2 and 4. As a consequence, candidate 7 can be eliminated from cells $(1, 4)$, $(6, 4)$, $(9, 2)$, and $(9, 4)$. Then completion can be attained by FN.

By $FNBT^2$, we could even reach state $(46, 81)$. Then the x-wing described above would lead to the elimination of candidate 7 from cells $(1, 4)$, $(6, 4)$, and $(9, 2)$, and completion would only require F.

sdk 140 By FNB, we reach state $(53, 70)$. Then in rows 3 and 7, candidate 2 is restricted to columns 3 and 7 and can therefore be eliminated from cells $(2, 7)$, $(5, 7)$, and $(8, 3)$. Then completion can be attained by FN.

sdk 141 By $FNBT^2$ (one hidden pair), we reach state $(29, 179)$. Then in rows 5 and 8, candidate 6 is restricted to columns 3 and 4. Therefore, candidate 6 can be eliminated from cells $(6, 4)$, $(7, 3)$, and $(9, 3)$. Completion is now possible by FNB.

sdk 142 By $FNBT^2$, we reach state $(45, 112)$. We now have even two "x-wings". In rows 1 and 9, candidate 4 is restricted to columns 1 and 7 and can therefore be eliminated from cells $(2, 7)$, $(7, 1)$, $(7, 7)$, and $(8, 7)$. And in the same rows, candidate 5 is restricted to columns 3 and 7, whence it can be eliminated from the 8 cells $(2, 3)$, $(2, 7)$, $(3, 3)$, $(3, 7)$, $(7, 3)$, $(7, 3)$, $(8, 3)$, $(8, 7)$. Completion then only requires F.

sdk 143 By $FNBT_4^2$, we reach state $(46, 102)$. Then in rows 3 and 5, candidate 1 is restricted to columns 1 and 9. It can therefore be eliminated from cells $(7, 9)$, $(8, 1)$, and $(8, 9)$. Then completion can be attained by FN alone.

sdk 144 By $FNBT_3$, we reach state $(54, 67)$. Then in rows 3 and 4, candidate 1 is restricted to columns 5 and 7. It can therefore be eliminated from cells $(1, 7)$, $(5, 7)$, and $(8, 7)$. Completion then only requires FN.

9 Miscellany

9.1 A kind of meta rule

My collegue Hans Egli, a mathematician from Zürich, drew my attention to a possibility of sudoku completion which is neither based on constraint propagation nor the usual way of trial and error (backtracking). Starting out from the sudoku on the left, by using W and Y as well as $FNBT$, we can get the sudoku on the right. But now, constraint propagation does not lead any further.

	5			**4**			**6**	
			9	**3**	**8**			
		8				**3**		
	7	**4**		**9**		**5**	**1**	
		2				**8**		
			2	**6**	**3**			
	1			**7**			**4**	

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2 4 7 9	3 8	1 7 9	1 6 7	1 2 5	1 2 5 6 7	1 2 4 7 9	3 8	1 2 7 9
2 7 8 9	**5**	1 3 7 9	1 7	**4**	1 2 7	1 2 7 9	**6**	3 8
2 4 6 7	2 4	1 6 7	**9**	**3**	**8**	1 2 4 7	2 5 7	1 2 5 7
1 5	6 9	**8**	1 4 6 7	1 2 5	1 2 4 5 6 7	**3**	2 7 9	2 4 6 7 9
3	**7**	**4**	**8**	**9**	2 6	**5**	**1**	2 6
1 5	6 9	**2**	**3**	1 5	4 6 7	**8**	7 9	4 6 7 9
4 7 9	4 8	5 7 9	**2**	**6**	**3**	1 7 9	5 7 8 9	1 5 7 9
2 6 8	**1**	3 6	**5**	**7**	**9**	2 6	**4**	3 8
2 6 7 9	2 3	5 6 7 9	1 4	**8**	1 4	6 7 9	2 3 5 7 9	2 5 7 9

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In column 5, one of the cells $(1,5)$ and $(4,5)$ finally has to get the digit 2. If we assume this to be cell $(1,5)$, we note the following immediate consequences:

- The candidate set in cell $(4,5)$ reduces to $\{1,5\}$.
- We therefore have an open pair $\{1,5\}$ in row 4, whence candidates 1 and 5 can be eliminated from cell $(4,6)$.

Now in each of the 4 cells $(4,1)$, $(4,5)$, $(6,1)$, and $(6,5)$, we have the same candidate set, which is $\{1,5\}$. In the boxes, rows, and columns which contain theses cells, candidates 1 and 5 do not appear outside of these 4 cells. If in any completion, we interchange 1 and 5 in these 4 cells, the other cells would not be affected. Thus if we had any completion at all, we would have at least two, contradicting the assertion that the sudoku pattern is a sudoku, that is, has exactly one completion. As a consequence, in column 5, we put the 2 into cell $(4,5)$.

Completion now is not simple; but it is possible by constraint propagation. Again, W and Y as well as $FNBT$ are needed. Trial and error usually means propagation to a contradiction, meaning that there is no completion. Here it means propagation to a point where there is more than one completion.

9.2 About diabolical sudoku problems

Sudokus are frequently grouped according to difficulty. It is, however, not advisable to let oneself be overly impressed by terms like "diabolical". In Mepham[8], Mepham[9], and Mepham[10], for instance, there are a total of 36 sudokus described as "diabolical". But they are of widely different degrees of difficulty:

Rules sufficient	Number of sudokus	%
FNB	3	8.3
FNT_2^0	1	2.8
$FNBT_2^0$	2	5.6
$FNBT_2^2 + X$ or $FNBT_2^2 + Y$	4	11.1
$FNB + X$	2	5.6
$FNBT_3^2 + Y$	8	22.2
$FNBT_3^0 + W_2$	1	2.8
$FNBT_3^2 + W_3 + Y$	2	5.6
Trial and error	13	36.1

Of the 36 sudokus, 6 (i. e. 16.7 %) can be completed by $FNBT$ alone.

We find a similar situation in Die Zeit/Handelsblatt[11]. In the last 36 sudokus, all described as "teuflisch schwer" (devilishly difficult), the methods required are as follows:

Rules sufficient	Number of sudokus	%
FN	4	11.1
FNB or FNT_2^2	3	8.3
FNB	3	8.3
FNT_2^2	4	11.1
$FNBT_2^2 + X$ or $FNBT_2^2 + Y$	5	13.9
$FNBT_2^2 + Y$	12	33.3
$FNB + X_2$	2	5.6
Trial and error	3	8.3

The number of relatively easy problems is considerable: 14 of the 36 sudokus (i. e. 38.9 %) can be completed by $FNBT$ alone, and none of them require both rules B and T. Thereby, open and hidden pairs suffice.

In 10 cases, one single x- or y-chain suffices for completion; only 2 sudokus require for completion more than 4 chains.

10 More on Pair Sequences

10.1 Y-sequences and domino sequences

In order to apply rule 6 (Y) in the course of sudoku completion, we have to spot y-chains, and therefore y-sequences. We can, of course, determine by trial and error, whether a given pair sequence is a y-sequence, i.e., whether or not it can be written in the form

$$(\{c_0, c_1\},\ \{c_1, c_2\},\ \ldots,\ \{c_{n-2}, c_{n-1}\},\ \{c_{n-1}, c_0\}).$$

Trial and error show, for example, that

$$(\{5,7\},\ \{4,5\},\ \{4,5\}, \{2,5\},\ \{1,2\},\ \{1,3\},\ \{1,3\},\ \{1,4\},\ \{2,4\},\ \{2,7\})$$

is a y-sequence, while

$$(\{5,7\},\ \{4,5\},\ \{4,5\},\ \{2,5\},\ \{1,2\},\ \{1,3\},\ \{1,4\},\ \{2,4\},\ \{2,7\}) \tag{2}$$

is not. However, the y-sequences give rise to many interesting questions.

If we want to have a closer look at y-sequences, it is mathematically recommendable to direct our attention to the slightly more general domino sequences. The reason lies in the fact that subsequences of y-sequences are not, in general, y-sequences themselves, while they are always domino sequences. Remember that a pair sequence Π is a domino sequence, if there is a strictly inhomogeneous sequence $(c_0, \ldots, c_n)$ such that

$$\Pi = (\{c_0, c_1\},\ \{c_1, c_2\},\ \ldots,\ \{c_{n-1}, c_n\}).$$

The sequence $(c_0, \ldots, c_n)$ we call a *spine* of Π (cf. definition 17).

One first condition is obvious. Domino sequences are necessarily *contiguous* in the following sense:

Definition 18 (Contiguous pair sequence) *We say that a pair sequence is* contiguous*, if any two consecutive pairs have at least one element in common.*

A domino sequence has one ore more spines. In fact, the number is at most 2:

Lemma 10.1 (Number of spines) *(i) If $(c_0, c_1, \ldots, c_n)$ and $(c_0, c_1^*, \ldots, c_n^*)$ are both spines of one and the same pair sequence, then $c_i = c_i^*$ for $i = 1, \ldots, n$.*

(ii) Any given pair sequence has at most two distinct spines

Proof: (i) By induction on n. Let $\Pi = (\pi_1, \ldots, \pi_n)$ be the given pair sequence. If $n = 1$, the spines are (c_0, c_1) and (c_0, c_1^*). Π consists of just one pair π_1, $\pi_1 = \{c_0, c_1\} = \{c_0, c_1^*\}$. Therefore, $c_1 = c_1^*$. If $n \geq 1$, $(c_0, c_1, \ldots, c_{n-1})$ and $(c_0, c_1^*, \ldots, c_{n-1}^*)$ are both spines of $(\pi_1, \ldots, \pi_{n-1})$. By induction hypothesis, $c_i = c_i^*$ for $i = 1, \ldots, n-1$. Therefore, $\pi_n = \{c_{n-1}, c_n\} = \{c_{n-1}^*, c_n^*\} = \{c_{n-1}, c_n^*\}$, whence $c_n = c_n^*$.

(ii) Let $\pi_1 = \{a_1, b_1\}$. Any spine starts with a_1 or with b_1. By (i), there is in both cases at most one distinct spine. Q.E.D.

10.2 Strictly inhomogeneous pair sequences

We repeat that by a strictly inhomogeneous sequence, we understand a sequence in which any two consecutive elements (or pairs) are distinct (definition 15).
In a contiguous, strictly inhomogeneous pair sequence, every two consecutive pairs have exactly one common element. We therefore define:

Definition 19 (Derivative; leading, trailing element) *For any contiguous, strictly inhomogeneous pair sequence* $\Pi = (\pi_1, \ldots, \pi_n)$*, where* $n \geq 2$*, the conditions*

$$\begin{cases} d_0 \in \pi_1, \quad d_0 \notin \pi_2; \\ d_i \in \pi_i, \quad d_i \in \pi_{i+1} \text{ for } i = 1, \ldots, n-1; \\ d_n \notin \pi_{n-1}, \quad d_n \in \pi_n \end{cases}$$

unambiquously define a sequence $(d_0, \ldots, d_n)$ *which we call the* derivative *of* Π*. Thereby,* $d_0 \notin \pi_2$*,* $d_1 \in \pi_2$*, and* $d_{n-1} \in \pi_{n-1}$*,* $d_n \notin \pi_{n-1}$ *imply*

$$d_0 \neq d_1, \quad d_{n-1} \neq d_n.$$

We call d_0 *the* leading*, and* d_n *the* trailing *element of* Π*.*

The above definition says that d_0 is the unique element of π_1 not in π_2, that $d_1, \ldots, d_{n-1}$ are the elements of $\pi_1, \ldots, \pi_{n-1}$ not contained in the next pair; and that d_n is the unique element of π_n not in π_{n-1}.

Example 10.1 *Here are three examples of contiguous, strictly inhomogeneous pair sequences and their derivatives:*

(i) $(\{1,2\}, \{1,3\})$: The derivative is $(2,1,3)$.

(ii) $(\{1,2\}, \{2,3\}, \{3,4\})$: The derivative is $(1,2,3,4)$.

(iii) $(\{1,2\}, \{1,3\}, \{1,4\})$: The derivative is $(2,1,1,4)$.

In the third example, the derivative is not strictly inhomogeneous.

Theorem 1 (Derivative) *Let* $\Pi = (\pi_1, \ldots, \pi_n)$ *be a contiguous and strictly inhomogeneous pair sequence. Then the following propositions hold:*

(i) If Π *is a domino sequence, then it has exactly one spine, and the spine coincides with the derivative.*

(ii) Π *is a domino sequence, if and only if the derivative is strictly inhomogeneous.*

Proof: (i) Assume that Π is a domino sequence, with spine $(c_0, \ldots, c_n)$. Then for $i = 1, \ldots, n-1$, $\pi_i = \{c_{i-1}, c_i\}$ and $\pi_{i+1} = \{c_i, c_{i+1}\}$, whence $c_i \in \pi_i$ and $c_i \in \pi_{i+1}$. By definition 19, $c_i = d_i$ for $i = 1, \ldots, n-1$, Therefore, $\pi_1 = \{c_0, c_1\} = \{c_0, d_1\}$, and

$\pi_n = \{c_{n-1}, c_n\} = \{d_{n-1}, c_n\}$. But by definition 19, $d_0 \in \pi_1$ and $d_0 \neq d_1$, whence $c_0 = d_0$. Analogously, $c_n = d_n$.
(ii) By (i), if Π is a domino sequence, the derivative is the one and only spine. By definition, the spine is strictly inhomogeneous. Conversely, let the derivative be strictly inhomogeneous. By definition 19, $d_{i-1}, d_i \in \pi_i$ for $i = 1, \ldots, n$. As now $d_{i-1} \neq d_i$, this becomes $\pi_i = \{d_{i-1}, d_i\}$. So $(d_0, \ldots, d_n)$ is a spine for Π, which makes Π a domino sequence. Q.E.D.
The pair sequence (2) contains the subsequence $(\{1,2\}, \{1,3\}, \{1,4\})$, which, having the derivative $(2,1,1,4)$, is not a domino sequence. Therefore (2) is not a domino sequence by lemma 7.2.

10.3 A decision method

Although it is always possible to check by trial and error, whether a given sequence is a a domino sequence, there are sometimes shorter ways. In the case of strictly inhomogeneous pair sequences, theorem 1 gives an answer. The following theorem is a further, negative, criterion:

Theorem 2 (Omitting subsequences) *If from a domino sequence containing a homogeneous subsequence of even length, this subsequence is omitted, the remaining sequence is still a domino sequence.*

Proof: If the homogeneous subsequence is located an the beginning or the end of the given domino sequence, the conclusion follows from lemma 7.2. As a subsequence of even length can be omitted by repeatedly omitting a subsequence of length 2, it remains to prove the lemma for the case where in the domino sequence

$$(\{c_0, c_1\}, \ldots, \{c_{i-1}, c_i\}, \{c_i, c_{i+1}\}, \{c_{i+1}, c_{i+2}\}, \{c_{i+2}, c_{i+3}\}, \ldots, \{c_{n-1}, c_n\})$$

$\{c_i, c_{i+1}\} = \{c_{i+1}, c_{i+2}\}$. Then $c_i = c_{i+2} = a$, and after the omission of $\{c_i, c_{i+1}\}$ and $\{c_{i+1}, c_{i+2}\}$, their remains the pair sequence

$$(\{c_0, c_1\}, \ldots, \{c_i, a\}, \{a, c_{i+3}\}, \ldots, \{c_{n-1}, c_n\}),$$

which is a domino sequence. Q.E.D.

By this lemma, we can immediately see that $(\{1,2\}, \{2,3\}, \{2,3\}, \{3,4\})$ is not a domino sequence. If it were, then $(\{1,2\}, \{3,4\})$ also would be. But this latter is not contiguous.

Unfortunately, the converse of theorem 2 does not hold, even if we demand the resulting sequence be contiguous. If the homogeneous sequence $(\{1,3\}, \{1,3\})$ is inserted into the domino sequence $(\{1,2\}, \{2,3\})$, the resulting sequence, $(\{1,2\}, \{1,3\}, \{1,3\}, \{2,3\})$, is not a domino sequence. It is not possible to arrange the pairs in the required order.

As spines are, by definition, strictly inhomogeneous sequences, the following theorem is obvious:

Theorem 3 (Homogeneous sequences) *If* Π *is homogeneous,* $\Pi = (\{a,b\}, \{a,b\}, \ldots, \{a,b\})$*, where* $a \neq b$*, then* Π *has exactly two spines. They are:*

$$(a, b, a, \ldots, b) \text{ and } (b, a, b, \ldots, a),$$

if Π *is of* even *length (consists of an even number of pairs), and*

$$(a, b, a, \ldots, a) \text{ and } (b, a, b, \ldots, b),$$

if Π *is of* odd *length.*

Lemma 10.2 (Fitting domino sequences) *If* Π *and* Π^* *are domino sequences, then their catenation* $\Pi\,\Pi^*$ *is a domino sequence, if and only if there are spines* $(c_0, \ldots, c_m)$ *and* $(c_0^*, \ldots, c_n^*)$ *of* Π *and* Π^**, respectively, such that* $c_m = c_0^*$*.*

Proof: Let $\Pi = (\pi_1, \ldots, \pi_m)$, and $\Pi^* = (\pi_1^*, \ldots, \pi_n^*)$. If the catenation $\Pi\,\Pi^*$ is a domino sequence, then it has a spine $(c_0, \ldots, c_{m+n})$. But now $(c_0, \ldots, c_m)$ is a spine of Π, and $(c_m, \ldots, c_{m+n})$ is a spine of Π^*.
Conversely, let $(c_0, \ldots, c_m)$ and $(c_0^*, \ldots, c_n^*)$ be spines of Π and Π^*, respectively, and let $c_m = c_0^*$. Then $(c_0, \ldots, c_m, c_1^*, \ldots, c_n^*)$ is a spine of $\Pi\,\Pi^*$. Q.E.D.

We return to example 10.1. If we split the pair sequence $(\{1,2\}, \{1,3\}, \{1,4\})$ into $\Pi = (\{1,2\}, \{1,3\})$ and $\Pi^* = (\{1,4\})$, then Π has spine $(2,1,3)$ (and none other). Π^* has exactly two spines, $(1,4)$ and $(4,1)$. As $3 \neq 1, 4$, $\Pi\,\Pi^*$ is not a domino sequence.

Rule 8 (Decision) *Let* Π *be any contiguous pair sequence. It can be split up into subsequences each of which is either strictly inhomogeneous, or homogeneous and of even length. By theorem 1, we can decide whether all the strictly inhomogeneous subsequences are domino sequences. Now there are two cases.*
Case I. They are not all domino sequences. Then by lemma 7.2, the given sequence is not a domino sequence.
Case II. All strictly inhomogeneous subsequences are domino sequences. Then replace all subsequences with one of their spines (one possibility for strictly inhomogeneous sequences, two for homogeneous ones). If spines can be made to fit in the sense of lemma 10.2, then the given pair sequence is a domino sequence; otherwise, it is not.

Example 10.2 *We return to the pair sequence (1) of example 7.2. It can be split up, for example, according to rule 8 as follows:*

(1) $(\{5,7\}, \{7,1\}, \{1,6\})$,
(2) $(\{1,6\}, \{6,1\})$,
(3) $(\{1,6\}, \{1,9\}, \{9,3\})$,
(4) $(\{3,9\}, \{9,7\})$,
(5) $(\{7,9\}, \{9,8\}, \{8,2\}, \{2,3\}, \{3,8\})$,
(6) $(\{8,3\}, \{3,1\}, \{1,4\}, \{4,8\}, \{8,5\})$.

Fitting spines are $(5,7,1,6)$, $(6,1,6)$, $(6,1,9,3)$, $(3,9,7)$, $(7,9,8,2,3,8)$, $(8,3,1,4,8,5)$. The second subsequence also has spine $(1,6,1)$, but this would not fit. All other spines are uniquely determined, the subsequences being strictly inhomogeneous. The result is a spine which begins and ends with 5; therefore, the pair sequence is not only a domino sequence, but a y-sequence.

10.4 A further approach: signatures and choice sequences

Definition 20 (Signature of a cell chain) *We call* (a, b) *a* signature *of a pair chain, if in every valid assignment, either* a *is assigned to the first cell, or* b *is assigned to the last cell.*

By a *valid* assignment, of course, we mean an assignment in which every two consecutive digits are distinct. A pair chain has signature (a, b), if and only if for any valid assignment, if the first cell is not assigned a, then the last is assigned b. Note that signatures are *ordered* pairs.

The following lemma enumerates some immediate consequences of the above definition:

Lemma 10.3

(i) A domino chain of type (c, e) *has signature* (c, e).

(ii) A pair chain is a y-chain with respect to candidate c, if and only if it has signature (c, c).

(iii) If a cell chain begins with a subchain of signature (a, b), *and the remaining subchain has signature* (b, c), *then the chain has signature* (a, c).

(iv) If a cell chain has signature (a, b), *then the reverse chain has signature* (b, a).

(v) A homogeneous pair chain with candidate sets $\{a, b\}$ *has signatures* (a, a) *and* (b, b), *if the number of edges is odd (hence the number of cells even), and* (a, b) *and* (b, a) *otherwise.*

Not that there is no contradiction of (v) with theorem 3. The latter mentions the number of pairs in a pair sequence. The former counts edges of a chain. The number of pairs (vertices, cells) is even, if and only if the number of edges is odd.

Part (v) is even true for chains with 0 edges, i.e. isolated cells: In the 0-edged chain $\{1, 2\}$, the only cell is at the same time first and final cell, and the signatures are $(1, 2)$ and $(2, 1)$.

We return to example 10.2. The 6 subsequences have signatures $(5, 6)$, $(6, 6)$, $(6, 3)$, $(3, 7$, $(7, 8)$, $(8, 5)$, respectively. By lemma 10.3 (iii), the full sequence has signature $(5, 5)$ and is therefore a y-sequence, with respect to candidate 5. We might also attribute to each cell its fitting signature, thereby making use of the remark on 0-edged chains immediately after lemma 10.3. We then get in turn the signatures (5,7), (7,1), (1,6), (6,1), (1,6), (6,1), (1,9), (9,3), (3,9), (9,7), (7,9), (9,8), (8,2), (2,3), (3,8), (8,3), (3,1), (1,4), (4,8), (8,5), which, by lemma 10.3 (iii), again reduce to (5,5).

10.5 Pair sequences

Remember that we always speak of cell chains with a definite candidate list in mind. Therefore, to each cell chain corresponds a unique pair sequence, and to every valid assignment of the chain, there corresponds a sequence of candidates in which any two consecutive elements are distinct. This leads us to define:

Definition 21 (Choice sequence) *We call* $(c_1, \ldots, c_n)$ *a* choice sequence *of the pair sequence* $\Pi = (\pi_1, \ldots, \pi_n)$, *if* $c_i \in \pi_i$ *for* $i = 1, \ldots, n$.

Lemma 10.4 *For any pair sequence* $\Pi = (\pi_1, \ldots, \pi_n)$, *there exist two strictly inhomogeneous choice sequences* $(c_1, \ldots, c_n)$ *and* $(d_1, \ldots, d_n)$ *such that* $c_i \neq d_i$ *for* $i = 1, \ldots n$.

PROOF: By induction on n. $n = 1$: $\pi_1 = \{a_1, b_1\}$. We let $c_1 = a_1$, $d_1 = b_1$. $n > 1$: By induction hypothesis, there are strictly inhomogeneous choice sequences $(c_1, \ldots, c_{n-1})$ and $(d_1, \ldots, d_{n-1})$ for $\Pi = (\pi_1, \ldots, \pi_{n-1})$ such that $c_i \neq d_i$ for $i = 1, \ldots, n-1$. Case (I): $\{c_{n-1}, d_{n-1}\} \cap \pi_n = \emptyset$, where $\pi_n = \{a_n, b_n\}$. Let $c_n = a_n$ and $d_n = b_n$. Case (II): $\{a_{n-1}, b_{n-1}\} = \{c_{n-1}, d_{n-1}\}$ has (at least) one common element with $\{a_n, b_n\}$. We may, without loss of generality, assume $c_{n-1} = a_n$. Then $c_{n-1} \neq b_n$ and $d_{n-1} \neq a_n$. Therefore we let $c_n = b_n$ and $d_n = a_n$. Q.E.D.

The term *signature* can readily be applied to pair sequences:

Definition 22 (Signature of a pair sequence) *We call* (a, b) *a* signature *of a pair sequence* Π, *if every strictly inhomogeneous choice sequence of* Π *starts with* a *or ends with* b. *We denote the set of signatures of* Π *by* $\Sigma(\Pi)$.

(By *or*, we always mean *and/or*.)

Lemma 10.5 *If* (a, b) *is a signature of* $\Pi = (\pi_1, \ldots, \pi_n)$, *then* $a \in \pi_1$ *and* $b \in \pi_n$.

PROOF: By Lemma 10.4, there are strictly inhomogeneous pair sequences (1) $(c_1, \ldots, c_n)$ and (2) $(d_1, \ldots, d_n)$ of Π such that $c_i \neq d_i$ for $i = 1, \ldots, n$. Now assume that $a \notin \pi_1$. Then $a \neq c_1$, $a \neq d_1$. Because (a, b) supposedly is a signature, this implies $c_n = d_n = b$, contradicting $c_n \neq d_n$. Analogously, the assumption $b \notin \pi_n$ leads to the contradiction $c_1 = d_1 = a$. Q.E.D.

This lemma implies that any signature of pair sequence $\Pi = (\{a_1, b_1\}, \ldots, \{a_n, b_n\})$ is one of the 4 pairs (a_1, a_n), (a_1, b_n), (b_1, a_n), (b_1, b_n). Each signature excludes a set of strictly inhomogeneous choice sequences according to their first and last element. The signature (a_1, b_n), for instance, excludes the choice sequences beginning with b_1 and ending on a_n.

Example 10.3

1. $\Pi = (\{1,2\}, \{1,2\}, \{1,2\}, \{1,2\})$. There are exactly 2 strictly inhomogeneous choice sequences: $(1,2,1,2)$, and $(2,1,2,1)$. Π has exactly 2 signatures: $\Sigma = \{(1,1),(2,2)\}$.

2. $\Pi = (\{1,2\}, \{1,2\}, \{1,2\})$. There are exactly 2 strictly inhomogeneous choice sequences: $(1,2,1)$, and $(2,1,2)$. Π has exactly 2 signatures: $\Sigma = \{(1,2),(2,1)\}$.

3. $\Pi = (\{1,2\}, \{2,3\}, \{3,4\})$. The strictly inhomogeneous choice sequences are: $(1,2,3)$, $(1,2,4)$, $(1,3,4)$, $(2,3,4)$. There is just one signature: $\Sigma = \{(1,4)\}$.

4. $\Pi = (\{1,2\}, \{1,3\}, \{1,4\})$. There are 5 strictly inhomogeneous choice sequences (c_1, c_2, c_3), and for (c_1, c_3), all 4 possibilities do occur: $(1,1)$, $(1,4)$, $(2,1)$, $(2,4)$. Therefore, there is no signature: $\Sigma = \emptyset$.

Lemma 10.6 *If (a,b) is a signature of $\Pi = (\pi_1, \ldots, \pi_n)$, then there are strictly inhomogeneous choice sequences (1) $(c_1, \ldots, b)$ such that $c_1 \neq a$ und (2) $(a, \ldots, c_n)$ such that $c_n \neq b$.*

Proof: By Lemma 10.5, $a \in \pi_1$ and $b \in \pi_n$. Let c_1 be the element of π_1 which is unequal to a. By 10.4, there exists a strictly inhomogeneous choice sequence $(c_1, \ldots, c_n)$. As (a,b) is a signature of Π, and $c_1 \neq a$, by definition $c_n = b$. The same argument, applied to the reverse pair sequence, shows that (2) exists. Q.E.D.

Lemma 10.7 *We denote the converse sequence of pair sequence Π by Π^*, and the catenation of pair sequences Π_1 and Π_2 by $\Pi_1\,\Pi_2$. By $||\Sigma(\Pi)||$ we denote the cardinality of the set $\Sigma(\Pi)$, i.e. the number of signatures of pair sequence Π. Then*

(i) $||\Sigma(\Pi^*)|| = ||\Sigma(\Pi)||$.

(ii) $||\Sigma(\Pi_1\,\Pi_2)|| \leq ||\Sigma(\Pi_2)||$.

(iii) $||\Sigma(\Pi_1\,\Pi\,\Pi_2)|| \leq ||\Sigma(\Pi)||$.

Proof: (i) By lemma 10.3, if (a,b) is a signature of Π, then (b,a) is one of Π^*. The converse also holds, as $\Pi^{**} = \Pi$.
(ii) If $\Pi_1\,\Pi_2$ is not contiguous, then $||\Sigma(\Pi_1\,\Pi_2)|| = 0$. Otherwise, we let $\Pi_1 = (\pi_1, \ldots, \pi_k)$, and $\Pi_2 = (\{a_{k+1}, b_{k+1}\}, \ldots, \{a_n, a_n\})$. By lemma 10.4, there are strictly inhomogeneous choice sequences $(c_1, \ldots, c_k)$ and $(d_1, \ldots, d_k)$ of Π_1 such that $c_i \neq d_i$ for $i = 1, \ldots, k$. Now let $(a_{k+1}, \ldots, x)$ and $(b_{k+1}, \ldots, y)$ be any two strictly inhomogeneous choice sequences of Π_2 $(x, y \in \{a,b\})$. Since $\Pi_1\,\Pi_2$ is contiguous, $\{c_k, d_k\}$ and $\{a_{k+1}, b_{k+1}\}$ have a common element. We may assume $a_{k+1} = c_k$. Then $b_{k+1} \neq c_k$ and $a_{k+1} \neq d_k$. Therefore, $(d_1, \ldots, d_k, a_{k+1}, \ldots, x)$ and $(c_1, \ldots, c_k, b_{k+1}, \ldots, y)$ are strictly inhomogeneous choice sequences of $\Pi_1\,\Pi_2$ of types (with respect to first and last element) $(d_1, \ldots, x)$ and $(c_1, \ldots, y)$, respectively. As there are at least as many types for $\Pi_1\,\Pi_2$, there are at most as many signatures.
(iii) Using (i) and (ii), we get $||\Sigma(\Pi_1\,\Pi_2)|| = ||\Sigma((\Pi_1\,\Pi_2)^*)|| = ||\Sigma(\Pi_2^*\Pi_1^*)|| \leq ||\Sigma(\Pi_1^*)|| = ||\Sigma(\Pi_1)||$. Therefore, $||\Sigma(\Pi_1\Pi\Pi_2)|| \leq ||\Sigma(\Pi_1\Pi)|| \leq ||\Sigma(\Pi)||$ Q.E.D.

Theorem 4 (Domino sequences) *A pair sequence has a signature, if and only if it is a domino sequence. It has exactly two signatures, if it is homogeneous, and exactly one otherwise.*

This theorem is a consequence of the following lemma:

Lemma 10.8

(i) If a pair sequence is not contiguous, it has no signature.

(ii) Every pair sequence has at most 2 signatures.

(iii) Every pair sequence which is not homogeneous has at most one signature.

(iv) A pair sequence has a signature if and only if is a domino sequence.

(v) A pair sequence has 2 signatures if and only if it is homogeneous.

PROOF: (i) If pair sequence Π is not contiguous, then $\Pi = \Pi_1 \Pi_2$ for some $\Pi_1 = (\pi_1, \ldots, \pi_k)$ and $\Pi_1 = (\pi_{k+1}, \ldots, \pi_n)$ such that π_k and π_{k+1} have no common element. By lemma 10.4, there are strictly inhomogeneous choice sequences $(c_1, \ldots, c_k)$ and $(d_1, \ldots, d_k)$ of Π_1 such that $c_i \neq d_i$ for $i = 1, \ldots, k$; and strictly inhomogeneous choice sequences $(c_{k+1}, \ldots, c_n)$ and $(d_{k+1}, \ldots, d_n)$ of Π_2 such that $c_i \neq d_i$ for $i = k+1, \ldots, n$. By catenation, we get strictly inhomogeneous choice sequences for Π of all four types (with respect to first and last element). Therefore, there is no signature.

(ii) Let the given pair sequence be $\Pi = (\{a_1, b_1\}, \ldots, \{a_n, b_n\})$. Then for any choice sequence $(c_1, \ldots, c_n)$, for (c_1, c_n) there are 4 possibilities: (a_1, a_n), (a_1, b_n), (a_n, b_1), (a_n, a_n). Every signature excludes one of these. But by Lemma 10.4, there have to remain at least 2 possibilities.

(iii) If Π is not homogeneous, then $\Pi = \Pi_1 \Pi_0 \Pi_2$ for some $\Pi_0 = (\{a, b\}, \{c, d\})$ where $\{a, b\} \neq \{c, d\}$. We may assume that $d \notin \{a, b\}$ and $c \neq b$. Then (a, d), (b, d), and (b, c) are all strictly inhomogeneous choice sequences, and their remains at most one possible signature, (a, c). By part (iii) of lemma10.7, Π also has at most one signature.

(iv) If Π is the domino sequence $(\{c_0, c_1\}, \ldots, \{c_{n-1}, c_n\})$, it has the signature (c_0, c_n).

It remains to prove the converse. By (ii), there are at most 2 signatures. If there are 2, then by (iii), Π is homogeneous, and therefore is a domino sequence. We now assume that Π has exactly one signature, and let $\Pi = (\{a_1, b_1\}, \ldots, \{a_n, b_n\})$, and $\Pi_0 = (\{a_1, b_1\}, \ldots, \{a_{n-1}, b_{n-1}\})$. ($\Pi_0$ is Π without the last element.) By lemma 10.7, $||\Sigma(\Pi)|| \leq ||\Sigma(\Pi_0)||$, whence $||\Sigma(\Pi_0)|| \geq 1$. By induction hypothesis, Π_0 is a domino sequence. We now have two cases, depending on whether Π_0 is homogeneous or not.

Case I. If Π_0 is homogeneous, then $\Pi_0 = (\{a, b\}, \ldots, \{a, b\})$, and we may assume that $a_n = a$, since Π is contiguous. Now if Π_0 is of even length, we have $\Pi = (\{a, b\}, \{b, a\}), \ldots, \{b, a\}), \{a, b_n\})$. Otherwise, we get $\Pi = (\{b, a\}), \{a, b\}, \ldots, \{b, a\}), \{a, b_n\})$. In both cases, Π is a domino sequence.

Case II. If Π_0 is not homogeneous, by induction hypothesis, $\Pi = (\{c_0, c_1\}, \{c_1, c_2\}, \ldots, \{c_{n-2}, c_{n-1}\}), \{a_n, b_n\})$, where $\Sigma(\Pi_0) = \{(c_0, c_{n-1})\}$. Therefore, Π_0 has strictly inhomogeneous choice sequences of type $(c_0, \ldots, c_{n-1})$ as well of type $(c_1, \ldots, c_{n-1})$. (Only type $(c_1, \ldots, c_{n-2})$ is excluded by the signature.) If $c_{n-1} \in \{a_n, b_n\}$, we are done. Finally, we show that there is no other possibility. For assume that $c_{n-1} \notin \{a_n, b_n\}$. As Π is contiguous, this would lead to $c_{n-2} \in \{a_n, b_n\}$, and we then may assume $a_n = c_{n-2}$. Then $c_{n-2} \neq c_{n-1}$, $b_n \neq c_{n-1}$, so that $(c_0, \ldots, c_{n-1}, c_{n-2})$, $(c_0, \ldots, c_{n-1}, b_n)$, $(c_1, \ldots, c_{n-1}, c_{n-2})$, $(c_1, \ldots, c_{n-1}, b_n)$ would all be possible types of strictly inhomogeneous choice sequences for Π, contradicting the assumption that Π has a signature.

(v) If Π is homogeneous, $\Pi = (\{a, b\}, \dots, \{a, b\})$. If the length is even, we can write $\Pi = (\{a, b\}, \{b, a\}, \dots, \{b, a\})$ and $\Pi = (\{b, a\}, \{a, b\}, \dots, \{a, b\})$, and therefore Π has the signatures (a, a) and (b, b). If the length is odd, then we can write $\Pi = (\{a, b\}, \{b, a\}, \dots, \{a, b\})$ and $\Pi = (\{b, a\}, \{a, b\}, \dots, \{b, a\})$, and therefore Π has the signatures (a, b) and (b, a). The converse holds by (iii).
Q.E.D.

We conclude with one last lemma:

Lemma 10.9 *If* Π_1 *has signature* (a, b)*, and* $\Pi_1 \Pi_2$ *has signature* (a, c)*, then* Π_2 *has signature* (b, c)*.*

PROOF: Assume that (b, c) is not a signature of Π_2. Then for Π_2, there is a strictly inhomogneous choice sequence $(y, \dots, z)$ such that $y \neq b$, $z \neq c$. But by Lemma 10.6, Π_1 has a strictly inhomogeneous choice sequence $(x, \dots, b)$ such that $x \neq a$. Therefore, $(x, \dots, b, y, \dots, z)$ is a strictly inhomogeneous choice sequence of $\Pi_1 \Pi_2$ such that $x \neq a$ and $z \neq c$. This contradicts the assumption that Π_1, Π_2 has signature (a, c). Q.E.D.

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